From “Democracy” to Dictatorship

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This essay describes how a “democratic” voting can give a dictatorial outcome. In essence, choosing representatives in political systems could be understand as a coarse-graining process or decimation of opinions. Therefore, ideas and techniques from real-space renormalisation group can be used. This is a good example of the application of statistical physics to politics.

I. INTRODUCTION

Can we apply statistical physics to politics? The answer is yes. In these days, no one doubts that the modern statistical physics is a powerful tool to study many complex non-living systems. However, one may still wonder how statistical physics can been used to study the social system. After all, this is a non-living system. And the richness of behavior involved in human situations creates a great deal of complexity (1). Surprisingly, the application of statistical physics to social behavior has been tried for decades. And the field of so-called **socialphysics** has drawn a growing interest from physics, mostly theoreticians (2). In particular, models to describe the process of strikes (3), the group decision thinking (4), collective choice dynamics (5) and even the power-law in elections (6) have been presented.

Why can we do that? A naive reason is that the process of going in parallel from one atom and one human to respectively several atoms in bulk and a social group has much in common. A deep reason may be the abstract and general nature of the statistical physics framework (1), *e.g.* the fundamental concepts of **universality** and **irrelevant variables** in **critical phenomenon** and the **renormalisation group** (**RG**) theory. These concepts enable us to describe phase transitions in terms of only a small finite number of **universality class**, even though the number of physics systems undergoing phase transitions is almost infinite. Moreover, as announced by some physicists, the renormalisation group is one of the most profound discoveries in science, which is a theory about theories (7). It will not be a big surprise if it can be applied to fields outside of physics.

In this term essay, I try to review one of the applications of statistical physics to politics, i.e. the voting problem, which is one of most important and active subjects in the field of socialphysics. The voting problem is very important to politics, too. Because, we believe, from a common sense of politics, that a "democratic election" is the very first step to a "democratic regime" [1]. I focus on the voting problems in a hierarchical organizations made of small committees. The nation-wide general election is not discussed here.

According to the voting theory, the majority rule is almost the best voting rule [2]. *How can then dictatorship result from it?* Actually, such a social paradox was indeed found to result from the use of majority rule voting (3; 10–12). This issue was studied with the so-called Galam model, which is named in honor of its inventor Serge Galam, the father of social-physics. The basic idea is easily understood. In political systems, choosing representatives could be understood as a **coarse-graining** transformation or **decimation** of opinions (13). Then it's not surprising that some ideas from real-space renormalisation group can be used to study such voting problems.

II. MODEL

Consider a population distributed among two political directions A and B with respective probabilities p_0 and $1-p_0$. People are then *randomly* selected from the overall population to form cells (*e.g.* small committees), each one containing r person. To initiate the hierarchy, the bottom level is constituted using the above r-size cells. Then each cell elects a representative, either an A-person or a B-person using a local majority rule. These elected people constitute the first hierarchical level of the organization called level-1. New cells are then formed *randomly* at level-1 from these elected people. They in turn elect new representatives to build level-2. This process is repeated again and again until we reach a single person at

[1] Democracy is not equivalent to a democratic election. Actually, a democratic election doesn't guarantee a democratic regime. Otherwise, Saddam Hussein will not be the dictator of Iraq for so many years. A democratic election is just the necessary condition to form a democratic regime. For a deep understanding of what democracy indeed means, see Ref.(8).

[2] The discomfort of social scientists with the impossibility to find a reasonable voting rule has been formalized by Arrow's celebrated impossibility theorem, see Ref. (9). In other words, the best voting rule doesn't exist.

the top of the hierarchy. See Fig.1.

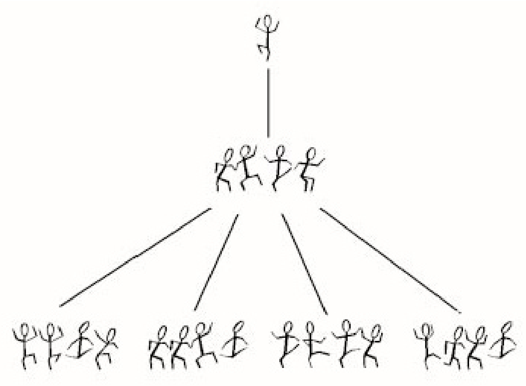

FIG. 1 The person at the top of the hierarchy is elected by the members of the middle level. These people are in turn elected by small groups from the main population.

It is worth to physically think about the model before we proceed. The system could be a political group, a firm, a network or a society. We assume that each member in the system does have an opinion about the political direction. For simplicity, we allow only two political direction A and B, just like in the Ising model spin could be either up or down. In physics the coarse-grained degree of freedom is fictitious while here it is a real person. The RG is just a theoretical scheme and could be performed infinite times in the RG theory, while here each voting step is real and the number of the voting steps must be finite. Because most organizations have only a few levels, and always less than 10.

Having a probability p_0 to get an A-person in a *randomly* formed cell [3], the probability p_1 to have an A-person elected by the cell is

$$p_1 = R(p_0) \tag{1}$$

where $R(p_0)$ is the **voting function**, i.e. the **recursion relation**, which accounts for all cell configurations producing an A-majority. For odd sizes, there always exist a clear majority. However, when r is even, the case of tie is particular. In most social situations it is well admitted and even well understood that to change a policy requires a majority. Therefore, in case of a tie, things stay as they are. Note that this is a bias in favor of the ruling leadership, which is the **origin of the dictatorship** as we will see later.

[3] Note that new cells are always formed randomly. This is the most elusive point in the Galam model, I think, which distinguishes it from both the real space RG transformation and the real voting process. However, it is seen that without doing this, we cannot iterate the voting function again and again.

Then the recursion relation $R(p)$ is given by

$$R(p) = \sum_{l=m}^{r} \frac{r!}{l!(r-l)!} p^l (1-p)^{r-l} \tag{2}$$

where $m = (r+1)/2$ for odd values of r and $m = (r+2)/2$ for even values of r. After n iterations we have an n-level hierarchy with a probability $p_n = R(p_{n-1})$ to have an A-person at top leadership.

III. RESULT

A. fixed points and RG flow

The recursion relation $R(p)$ is a monotonically increasing function of p. It has two trivial stable **fixed points** (FP) $p^* = 0$ and $p^* = 1$, in between there exists a non-trivial unstable fixed point which is $p_r^* = 0.5$ for odd r. When r is even and $r > 2$ we have $0.5 < p_r^* \leq p_4^* \approx 0.77$ and $p_r^* \to 0.5$ as $r \to \infty$. $r = 2$ is a special case, which has no non-trivial fixed point. See Fig.2. The RG flows for $r = 3$ and 4 are sketched in Fig.3.

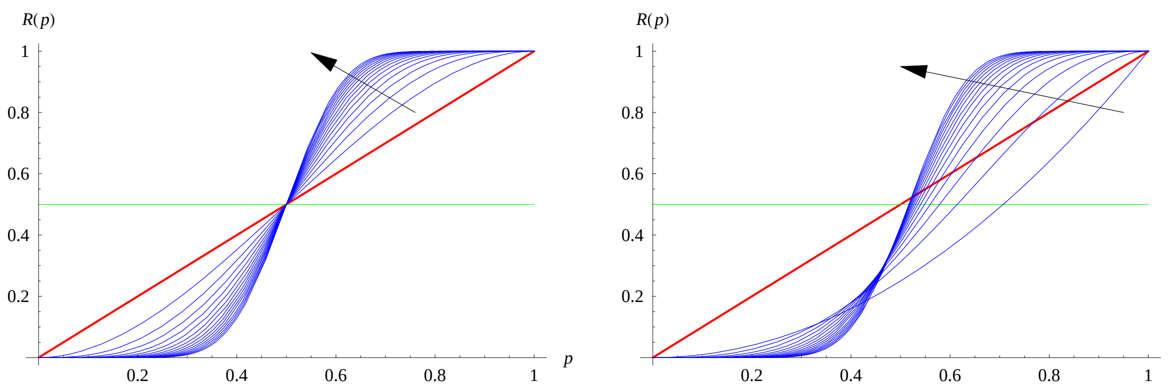

FIG. 2 Recursion relation for various values of r. $R(p) = p$ and $R(p) = 0.5$ are also plotted. (left): odd r=3,5,7,9...31; (right): even r=2,4,6,8...30. Arrows indicate the direction of r increasing. Non-trivial fixed point is just the point of intersection between $R(p)$ and the straight line $R(p) = p$. It's seen that for odd r, the intersection is fixed at 0.5 while for even r, the intersection is moving towards 0.5 with increasing r.

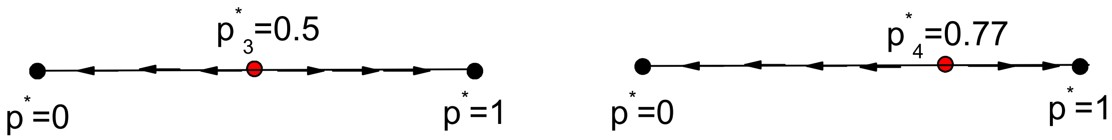

FIG. 3 RG flow. (left): r=3; (right): r=4.

B. dictatorship emerges

It's seen that when starting from $p_0 < p_r^*$ (or $p_0 > p_r^*$), the voting process generate a flow toward the stable fixed point $p^* = 0$ (or $p^* = 1$). Therefore, majority rule voting produces the self-elimination of any proportion of the A-tendency as long as $p_0 < p_r^*$. Of course, to complete the democratic self-elimination requires a sufficient number of voting levels.

For odd r, though the aggregating voting process eliminate a tendency, it stays **democratic**. This is because $R(p)$ is symmetrical and $p_r^* = 0.5$ in this case. It is the leading tendency (more than 50%) that after all gets the total leadership of the organization. For example, starting from $p_0 = 0.43 < p_r^* = 0.5$ with $r = 3$ we get successively $p_1 = 0.40$, $p_2 = 0.35$, $p_3 = 0.28$, $p_4 = 0.20$, $p_5 = 0.10$, $p_6 = 0.03$ down to $p_7 = 0.00$. Hence, 7 levels are sufficient to self-eliminate 43% (non-leading tendency) of the population with cell size $r = 3$.

For even r, things are different. The voting function becomes non-symmetrical and $p_r^* > 0.5$ for a finite r. Consequently, a majority can now be self-eliminated. For example, starting from $p_0 = 0.69 < p_r^* = 0.77$ with $r = 4$ we have the series $p_1 = 0.63$, $p_2 = 0.53$, $p_3 = 0.36$, $p_4 = 0.14$, $p_5 = 0.01$, and $p_6 = 0.00$. It shows that using a priori reasonable bias in favor of the B turns a majority rule democratic voting to a **dictatorial** outcome. Indeed to get to power the A must pass over 77% of support which is almost impossible in real world. This explains why it turns out very hard, if not impossible, to overthrow a ruling group.

C. critical number of levels: n_c

For given r, p_0, and $p_n = \epsilon$ with $0 < p_0 < p_r^*$ and ϵ a very small number, one can then calculate analytically the critical number of levels n_c at which $p_{n_c} = \epsilon$. This determines the level of confidence of the prediction to have no A elected. The basic idea is to linearize the voting function $p_n = R(p_{n-1})$ around the fixed points p^*:

$$p_n \simeq p^* + M(p_{n-1} - p^*) \tag{3}$$

where $M \equiv \frac{\mathrm{d}R}{\mathrm{d}p}|_{p^*}$ is the linearized voting function, corresponding to the **linearized RG transformation**.

For the stable FP $p^* = 0$, to the lowest order in p_{n-1}, one has

$$p_n \simeq \mu p_{n-1}^m \tag{4}$$

where $\mu = \frac{r!}{(m-1)!(r-m)!}$. Iterating Eq.4 n times gives

$$\mu^{1/(m-1)} p_n \simeq (\mu^{1/(m-1)} p_0)^{m^n} \tag{5}$$

from which one has

$$n_c = \frac{1}{\ln(m)} \ln[\frac{\ln(\mu^{1/(m-1)}\epsilon)}{\ln(\mu^{1/(m-1)} p_0)}]. \tag{6}$$

Note that Eq.5 holds only in the range that $\mu^{1/(m-1)} p_0 < 1$, otherwise p_n will be larger than p_0 which is obviously wrong for $p_0 < p_r^*$. For larger values of p_0, one has to perform the linearization around p_r^*:

$$p_n \simeq p_r^* + \lambda_r (p_{n-1} - p_r^*) \tag{7}$$

with $\lambda_r \equiv \frac{\mathrm{d}R}{\mathrm{d}p}|_{p_r^*}$. Iterating Eq.7 n times gives

$$p_n - p_r^* \simeq (p_0 - p_r^*) \lambda_r^n. \tag{8}$$

To optimize the range of validity of the above approximations in the whole range $0 < p_0 < p_r^*$, one can calculate the value $\bar{p}$ which minimizes the distance between Eq.4 and Eq.7. It is easy to get $\bar{p} = (\lambda_r / m\mu)^{1/(m-1)}$. If starting with a $p_0 > \bar{p}$ to go down to p_n, one has first to iterate Eq.7 up to $\bar{p}$ to get a number of iterations. Then one uses Eq.4 to go from

$\bar{p}$ down to $p_n = \epsilon$. The total number of iterations, i.e. the critical number of levels is then given by:

$$n_c = -\frac{\ln(p_r^* - p_0)}{\ln \lambda_r} + \frac{\ln(p_r^* - \bar{p})}{\ln \lambda_r} + \frac{1}{\ln(m)} \ln[\frac{\ln(\mu^{1/(m-1)}\epsilon)}{\ln(\mu^{1/(m-1)}\bar{p})}] \tag{9}$$

D. shrinking of the democratic region

We know that most social organizations have a fixed structure with a fixed number of hierarchical levels. Therefore, we invert the question on "how many levels are needed to eliminate a tendency" onto "given n levels what is the necessary overall support to get full power". From Eq.8 one has

$$p_0 \simeq p_r^* + (p_n - p_r^*)\lambda_r^{-n}. \tag{10}$$

Note that there are two critical thresholds. Put $p_n = 0$, one gets the **disappearance threshold**:

$$p_{d,r}^n = p_r^*(1 - \lambda_r^{-n}) \tag{11}$$

Put $p_n = 1$, one gets the **full-power threshold**:

$$p_{f,r}^n = p_{d,r}^n + \lambda_r^{-n} \tag{12}$$

A new region is then appeared: $p_{d,r}^n < p_0 < p_{f,r}^n$, in which A neither disappear totally nor get the full power. In other words, some democracy is prevailing since results of the election process are only probabilistic. No tendency is sure of winning, which makes alternating leadership a reality. For this reason, we call it the "**democratic region**". As seen from Eq.12, this region shrinks as **power law** λ_r^{-n}. See Fig.4. It's seen that for increasing r, not only the democratic region becomes smaller but also its shrinking accelerates when n increases. Ultimately, the democratic region shrinks into the unstable fixed point p_r^* for large n. In reality, this cannot happen since most organizations has only a few levels.

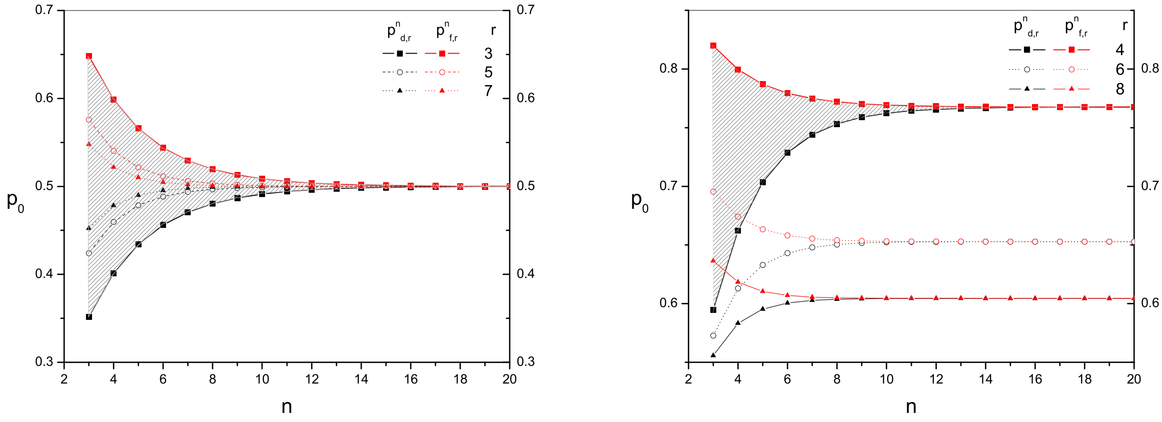

FIG. 4 Shrinking of the democratic region for various values of r. (left): odd r=3,5,7; (right): even r=4,6,8. The democratic regions for $r = 3$ and 4 are marked with shadow.

E. effect of non-voting people

In the simplest version of the Galam model, it's assumed that each person will vote. In reality, the existence of non-voting people, i.e. abstention due to sickness, apathy or other reasons, is a growing major feature in recent western country elections. *What is the effect of non-voting people?* It will be shown that the fixed point p_r^* will drastically shift away or even disappear.

Consider the simplest case: $r = 3$ which has $p_r^* = 0.5$ when there is no abstention. Assume that each person has a probability q of voting and $(1 - q)$ of non-voting by will, sickness, apathy or any other reason. We have to consider those cell configurations with two and one voting persons which can still produce an A-majority. According to the same principle "a majority is required to change things", both a non-voting cell (3 non-voting persons) and a tied 1A-1B cell(one non-voting person) will give an elected B representative. Therefore, the recursion relation is given by:

$$R(p) = q^3[p^3 + 3p^2(1-p)] + 3q^2(1-q)p^2 + 3q(1-q)^2p \qquad (13)$$

where the terms of RHS account for 3, 2 and 1 voting person case, respectively. Similarly, we can write the recursion relation for the $r = 4$ case:

$$R(p) = q^4[p^4 + 4p^3(1-p)] + q^3(1-q)[4p^3 + 12p^2(1-p)] + q^2(1-q)^26p^2 + q(1-q)^34p \quad (14)$$

As shown in Fig.5, with decreasing q, the two fixed points at $p^* = 1$ and p_r^* will move towards one another to merge at a certain value of $q = \bar{q}$. For $q < \bar{q}$, we see that only the fixed point $p^* = 0$ survives. For example, with $r = 3$, we have $\bar{q} \sim 0.81$, which means 19% of non-voting people are enough to make it impossible for the A group to win. Note that this democratic self-elimination is even much more stronger than that of the $r = 4$ no-abstention case with $q = 1.0$.

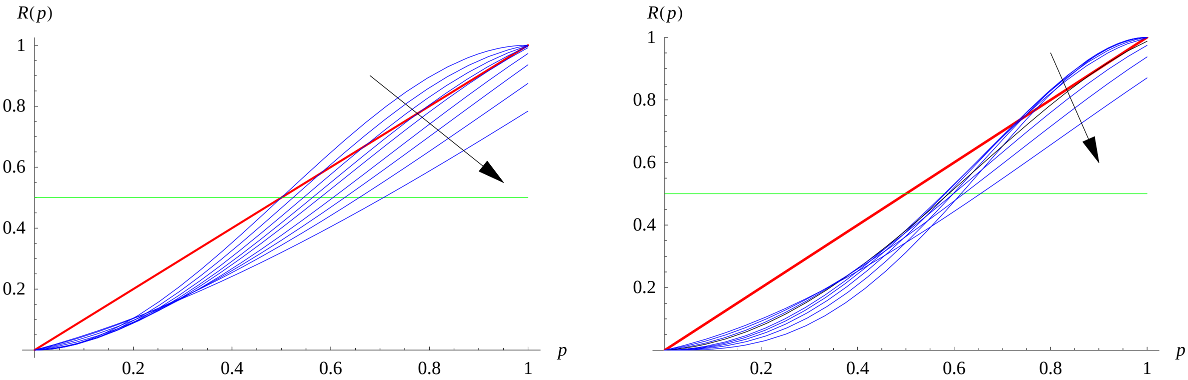

FIG. 5 Recursion relation for various values of q. Arrows indicate the direction of q decreasing. $R(p) = p$ and $R(p) = 0.5$ are also plotted. (left): odd r=3, q=1.0, 0.95, 0.9, 0.85, 0.8, 0.7 0.6, 0.5 and 0.4. The unstable fixed point p_3^* disappears at $q \sim 0.81$. (right): even r=4, q= 1.0, 0.95, 0.9, 0.85, 0.8, 0.665, 0.6, 0.5 and 0.4. The unstable fixed point p_4^* disappears at $q \sim 0.665$.

F. with more political directions

How about allowing more political directions? Physically, this corresponds to the change from the 2-state **Potts model** to the N-state Potts model. For simplicity, let's consider the case of three competing groups A, B and C with the $r = 3$ no-abstention case. We assume that in the case of a tie, i.e. 1A-1B-1C, we get A, B or C elected with probability α, β and $1 - \alpha - \beta$, respectively. The recursion relations are given by:

$$R(p_A) = p_A^3 + 3p_A^2(1 - p_A) + 6\alpha p_A p_B(1 - p_A - p_B) \tag{15}$$

$$R(p_B) = p_B^3 + 3p_B^2(1 - p_B) + 6\beta p_A p_B(1 - p_A - p_B) \tag{16}$$

$$R(p_C) = 1 - R(p_A) - R(p_B) \tag{17}$$

Then we can analysis the fix points and RG flow again. The main feature of the multi-group competition is that the bias results from parties agreement. Consequently, there are situations in which an initially very small minority tendency can finally win the election. For example, the two largest parties, say A and B, are usually hostile while the smallest one C would compromise with either one of them. Then the tie case (1A-1B-1C) gives a C elected, i.e. $\alpha = \beta = 0$. In other words, we need 2A or 2B to get A or B elected, respectively. Otherwise, C is elected. In such situation, the recursion relations for p_A and p_B are the same as for the model with two political directions. This means that the critical threshold to full power to A or B is 0.5. If both the initial supports for A and B are less than 0.5, then finally C get the full power. For more complicated cases with non-zero α and β, see Ref.(14).

IV. SUMMARY

In theory, for a two candidate election, the critical threshold to power is 50%. However, the usual existence of a reasonable bias makes it asymmetric and transform a democratic system in effect to a dictatorship. Also, the effect of the non-voting people has a drastic effect on the asymmetry of the threshold value. The Galam model is just a snapshot of reality, buy it grasps some essential and surprising mechanisms of majority rule voting, which could be used to explain the sudden collapse of some regimes in history. Dictatorship is just like a demon. It can result from a seemingly democratic process, e.g. the voting process. We have to elaborately design all the democratic processes to guarantee a democracy regime.

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