Physics 504: Statistical Mechanics and Kinetic Theory

HOMEWORK SHEET 5

Due 5pm Tue 9th April 2019 in the 504 box.

In this homework, you are asked to sketch certain functions. Please note that sketch means that you do NOT plot using mathematica or Matlab, but that you use your analysis skills to figure out the asymptotics of the functions, limiting forms, zeros, etc. In other words, estimate what the function looks like without plotting it.

Question 5–1.
Consider a quantum paramagnet of $N$ magnetic moments with $L = 0$ and $S = 1/2$.
(a) Calculate the contribution to the entropy from the magnetic moments, and sketch the result as a function of $1/x$, where $x = g\mu_B J H/k_B T$. By sketch, I mean figure out what the graph looks like, not plotting it on the computer!
(b) Now suppose that this system is at temperature $T_1$ and in a field $H_1$, and is thermally isolated from the surroundings. What happens to the temperature as the external field is reduced to zero?
(c) This phenomenon is a powerful experimental technique. What practical limitation does it have? Suggest how the method may be improved.

Question 5–2.
Photons are quantum mechanical bosons with unit spin, 2 helicity states (the quantum analogue of polarisation), whose number is not conserved. In this question, we will study the statistical mechanics of a gas of such particles, and derive the black body radiation spectrum from the “particle” point of view, as opposed to the “wave” point of view presented in lectures.
(a) What is the chemical potential of a gas of photons?
(b) Write down the partition function for a gas of photons with states labelled by momenta $k$ with energies $E_k$ and occupation numbers $n_k$.
(c) For photons, the dispersion relation is $E_k = \hbar c |k|$. Hence write down the free energy of a gas of photons in a volume $V$. You may use the fact that $\int_0^\infty x^2 \log(1 - e^{-x}) \, dx = -\frac{(2\pi)^4}{720}$.
(d) Hence calculate the average energy of the gas. You should recover the result obtained in lectures by another method.

Question 5–3.
Surface waves on liquid helium, at angular frequency $\omega$ and wavenumber $k = 2\pi/\lambda$ satisfy the following relation: $\omega^2 = (\sigma/\rho)k^3$ where $\sigma$ is the surface tension of liquid helium and $\rho$ is the density of liquid helium. Suppose that the liquid helium is at temperature $T$, and has surface area $A$. Show that the surface energy $U(T)$ — i.e. the energy of the surface waves — is given by

$$U(T) = CI \left( \frac{k_B T}{2\pi\hbar} \right)^\alpha$$

where $I = \int_0^\infty dx x^{4/3}/(e^x - 1) \approx 1.68$, and determine the coefficient $C$ and the value of the exponent $\alpha$.  

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Question 5–4.

For interacting systems, it may not be possible to solve the statistical mechanics exactly or even in any well-controlled approximation scheme. Sometimes, however, it is possible to make useful progress just using dimensional analysis.

(a) Consider a gas of classical particles of mass $m$ in a volume $V$ interacting via a pair potential $U(r)$. Sketch a typical form of $U(r)$. Suppose that for a particular class of substances, $U(r)$ has the form $U(r) = \epsilon u(r/\sigma)$. The meaning of this is that the energy scale is set by $\epsilon$ and the length scale by $\sigma$. For example, different gases might have different hard core radii $\sigma$ and binding energies $\epsilon$. Working in the canonical ensemble, show that all substances in this class have the same equation of state when expressed in suitably scaled variables. i.e. $p^* = \Pi(v^*, T^*)$, where starred quantities are scaled pressure, volume per particle and temperature, and $\Pi$ is a function that dimensional considerations alone cannot determine.

(b) Show that if there is a critical point for this class of fluids, then $p_c v_c / T_c$ is a constant independent of the particular fluid.

(c) The above theory works very well for gases like Neon, but significant departures are observed at low temperatures for gases like He and H$_2$. These are due to quantum effects. By dimensional analysis, show that the scaled equation of state should have the form

$$p^* = \Pi(v^*, T^*, \nu)$$

where $\Pi$ is some function that we cannot determine by these dimensional considerations alone, and $\nu = h/[\sigma \sqrt{m\epsilon}]$.

(d) Classically, the critical temperature $k_B T_c / \epsilon$ is a constant, but quantum mechanically, it depends on $\nu$. By plotting an appropriate graph (you may use a computer if you wish) estimate the critical temperature of the isotope $^3$He. You will need the following data.

<table>
<thead>
<tr>
<th></th>
<th>$^4$He</th>
<th>H$_2$</th>
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</thead>
<tbody>
<tr>
<td>$k_B T_c / \epsilon$</td>
<td>0.529</td>
<td>0.896</td>
</tr>
<tr>
<td>$\sigma / \text{Å}$</td>
<td>2.56</td>
<td>2.93</td>
</tr>
<tr>
<td>$\epsilon / k_B / \text{K}$</td>
<td>10.2</td>
<td>37</td>
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</tbody>
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You should justify what assumptions you find it necessary to make. The mass of the proton is $1.67 \times 10^{-27}$ Kg, and $k_B = 1.38 \times 10^{-23}$ JK$^{-1}$.

(e) In part (c), why did we use $\sigma$ for the length scale and not $v^{1/3}$ ($v \equiv V/N$)?