Lecture 3

Numerical methods and under-resolved computation
References for lecture 3


All my RG papers can be obtained in reprint form from

http://guava.physics.uiuc.edu/~nigel/articles/RG
GEOMETRIC INTERPRETATION

RG HAS A NON-PERTURBATIVE INTERPRETATION WHICH IS GEOMETRIC IN FUNCTION SPACE.

GENERAL IDEA

PERTURBATION THEORY DIVERGES
RENORMALISE MASS \( m(t) = Z^{-1}(\frac{1}{t^a}) m(0) \)
INTRODUCE ARBITRARY TIME SCALE \( t^* \)
SOLUTION CANNOT DEPEND UPON \( t^* \)

\[
\frac{du}{dt^*} = 0
\]

THIS IS A FIXED POINT CONDITION FOR A DYNAMICAL SYSTEM.

WE EXPLOIT THIS OBSERVATION BY CONSTRUCTING A DYNAMICAL SYSTEM...
**GEOMETRIC INTERPRETATION (2)**

**RG TRANSFORMATION $R_{b,y}$:**

$$u'(x,t_0) = R_{b,y}[u(x,t_0)]$$

**PERFORMS FOLLOWING STEPS**

1. **Evolve $u(x,t_0)$ forward in time using P.D.E. to $t_1 = bt_0$ ($b > 1$)**

2. **Rescale $x = b^\phi x'$. We will seek fixed points as $\phi$ is varied. For Barenblatt's Eqn, $\phi = 1/2$.**

3. **Rescale $u$:**

$$u'(0,t_0) = u(0,t_0)$$

$$u'(x,t_0) = z(b)u(b^{1/2}x, bt_0)$$

**This transformation is a (semi) group:**

$$R_{b_1} \circ R_{b_2} = R_{b_1b_2}$$

$$\Rightarrow \quad z(b) = b^y$$

$$y = \frac{d \log z}{d \log b}$$

**Fixed point $u^* = R_b[u^*]$ is similarity soln.**

$$u^*(x,t) = b^y u^*(b^{1/2}x, bt)$$

**Put $b = \frac{1}{\sqrt{t}}$**

$$u^*(x,t) = t^{-y} f\left(\frac{x}{\sqrt{t}}\right)$$
Convergence of exponent in modified porous medium equation with RG iteration number N
UNIVERSALITY AND NUMERICS

Any dynamical equation in basin of attraction of fixed point will exhibit same scaling features as the equation of interest.

.: can choose convenient dynamics

Example: Barenblatt's equation

Solve numerically using explicit scheme: discretisation $\Delta t, \Delta x$

As long as $\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$ [stability]

Anomalous dimension, scaling function not sensitive to $\Delta t, \Delta x$.

$\Rightarrow$ do not need to take continuum limit $\Delta t \to 0, \Delta x \to 0$

Example: "cell dynamic scheme" in approach to equilibrium, dendritic growth...
Front propagation
+
structural stability


Typically, near onset of instability, several modes exist and compete.

e.g. plane waves of varying orientations, wavelength etc.

\[ \partial_t \phi_i = D_i \Delta \phi_i + F_i(\phi_1, \ldots, \phi_N) \quad i = 1, \ldots, N \]

Amplitude of mode \( \phi_i \)  Diffusion constant  Reaction term

Stable  Unstable

Space
VELOCITY SELECTION IN FISHER-KPP EQN

FISHER (1937) AND KOLMOGOROV, PETROVSKY AND PISKUNOV (1937) STUDIED GENE SPREADING IN A POPULATION.

PROPOSED EQN:

\[ \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} + F(\varphi), \quad \varphi > 0 \]

\[ F = \varphi(1-\varphi), \quad \varphi(-\infty) = 1, \quad \varphi(+\infty) = 0 \]

STEADY STATE: \( e = x - ct \)

\[ \varphi'' + c \varphi' + F(\varphi) = 0 \]

ACCELERATION

FRICITION COEFFICIENT

FORCE \( = \frac{dV}{d\varphi} \)

TRAVELLING WAVE IS A TRAJECTORY STARTING AT \( \varphi = 1 \) AND STOPPING AT \( \varphi = 0 \).

MECHANICAL ANALOGUE: SPEED IS FRICTION.

ACCEPTABLE TRAJECTORIES: \( \varphi > 0 \)

\( c > \text{CRITICAL DAMPING} \)

\[ c > \hat{c} = 2\sqrt{F'(0)} \]
VELOCITY SELECTION ($\beta$)

STADY STATE ANALYSIS PREDICTS FAMILY OF STABLE TRAVELLING WAVES $C \geq \beta$.

BUT...

FOR ALL PHYSICAL INITIAL CONDITIONS, ONLY THE WAVE WITH $C = \beta$ IS FOUND, WHEN THE TIME-DEPENDENT PROBLEM IS SOLVED.

SELECTION PROBLEM: WE WOULD LIKE A SELECTION PRINCIPLE WHICH, A PRIORI, DISTINGUISHES OBSERVABLE FRONTS FROM UNOBSERVABLE ONES, WITHOUT HAVING TO SOLVE THE TIME-DEPENDENT PROBLEM.
Renormalization Group Algorithm

For travelling waves

\[ u(x, t) \]

**Eqn. of Motion**

**Step 1**

\[ u(x, t(1+b)) \]

**Shift**

**Step 2**

\[ u(x+vbt, (b+1)t) \]

**Seek Fixed Point**

\[ u(0, t) = u(d, t') \]

\[ t' = (1+b)t \]

\[ d = v_0(t'-t) \]

If \( v_0 \) is correct asymptotic velocity, shift returns wave to start.

In practice, use root finder to make vanish \( F(v) = u(0, t) - u(d, t') \)
\[ \frac{\partial u}{\partial t} = \frac{d^2 u}{dx^2} + u(1-u)(1+\gamma u), \quad -1 \leq \gamma < \infty \]

Solve by RG recursion to find travelling wave

KPP-Fisher Limit: \( \gamma = 0 \)
\( u(x,0) \sim e^{-q^2 x} \)

\( q \geq 1 \implies c = 2 \)

\( q < 1 \implies c = \frac{2 + \frac{1}{\gamma}}{\sqrt{2\gamma}} > 2 \)

\( \Delta x = 1.0, \Delta t = 0.05 \)

Speed transition as a function of initial condition

\( u(x,0) \sim e^{-q^2 x} \)

\( q > 1 \)

\[ c = 2, \quad -1 \leq \gamma < 2 \]

\[ c = \frac{2 + \gamma}{\sqrt{2\gamma}}, \quad \gamma > 2 \]

Speed transition as PDE is varied

\( \Delta x = 1.0, \Delta t = 0.05 \)

FIG. 2. The propagation velocity \( c \) as a function of decay rate \( q \) of initial conditions. The points determined by the numerical RG method are denoted by \( \bullet \). The continuous curve is the exact result.

FIG. 3. The propagation velocity \( c \) plotted as a function of \( \gamma \). The full curve represents the exact result, while data points determined by our numerical RG are denoted by \( \bullet \).
Structural Stability

Zeroth Law of Physics

Good Model of Reproducible Observable Physical Phenomena Must Give Structurally Stable Predictions.

Observable Predictions of Model Must Be Stable Against "Physically Small" Modifications of Model.

Example: Laws of Chemistry Not Dramatically Modified by Discovery of Top Quark.

Zeroth Law ⇒

1. We can do physics, without having to solve everything at once
2. It is hard to discover new laws (e.g. atoms only discovered recently)
STRUCTURAL STABILITY

STRUCTURAL STABILITY HYPOTHESIS

ONLY STRUCTURALLY STABLE SOLUTIONS
OF A MODEL REPRESENT REPRODUCIBLY
OBSERVABLE PHENOMENA.

POTENTIAL TAUTOLOGY

NEED TO SPECIFY "PHYSICALLY SMALL" CAREFULLY.

MODEL PREDICTION

P-large

P-small

USEFULNESS

WHAT ARE STRUCTURALLY STABLE
PREDICTIONS OF A MODEL? USE R.G.
**Small Perturbations**

\[ \partial_t \phi = \partial^2_x \phi + F(\phi) + \delta F \]

**C'-Small Perturbations**

Suppose \( \delta F \) is \( C' \)-small. This means \( \| \delta F \| \) and \( \| \delta F' \| \) are small in \([0,1]\).

The speed \( \dot{c} \) changes by a small amount.

Can be shown: model is structurally stable to \( C' \)-small perturbations.

**C-Small Perturbations**

The model is not stable to \( C \)-small \( \delta F \).

\[ \dot{c} = 2 \sqrt{F'(0)} \text{ can be made to increase without bound!} \]

This perturbation really has a large physical effect — not artifactual, e.g. dynamic on a fine wire.

Arbitrarily steep gradient, of small extent.
Physically small perturbations

We can restrict ourselves to the class of $C^0$-small perturbations for which the model is stable:

$p$-small perturbation is a $C^0$-small $\delta F$ for which $\sup_{\psi > 0} \frac{\delta F(\psi)}{\psi} < n$ where $n$ is a small positive number, with $n \to 0$ as $11\delta F11 \to 0$

Precise form of structural stability hypothesis:

Physically realisable solutions of Fisher's eqn are stable w.r.t. $p$-small perturbations.

This has predictive power!
P-LARGE PERTURBATIONS

\[ V \]

\[ \varphi \]

- \( F + \delta F \)
- \( F \)

Travelling wave is trajectory from \( \varphi = 1 \rightarrow \varphi = 0 \). The critical damping \( \hat{c} \) so that particle first stops at origin is greater than \( \hat{c}_{\text{original}} \).
Travelling wave is trajectory from

\( \varphi = 1 \rightarrow \varphi = 0 \). Only for one

Value of friction \((c)\) can

Particle end at rest at \( \varphi = 0 \)!

\[ c = c_{\text{original}} + O(\varepsilon) \]

Original minimum speed for unperturbed model

Goes to zero as \( \|\delta F\| \rightarrow 0 \)
IN RG LANGUAGE

\textit{p- small} \quad \textit{marginal perturbations}

\textit{C^0 - small (in general)} \quad \textit{relevant}

Front \quad \textit{similarity soln.} \quad \textit{fixed pt.}

\( \phi(x-ct) \xrightarrow{X=e^x, T=e^t} \Phi\left(\frac{x}{T^c}\right) \)

WE CAN USE RG IN TWO WAYS HERE:

\begin{enumerate}
  \item \textbf{Compute change} \( \delta c \) \textbf{to front speed} \( c \) \textbf{when} \( \text{p-small} \) \( \delta F \) \textbf{is present}.
  \item \textbf{Find} \( \delta c \) \textbf{such that} \( \delta c(z) \rightarrow 0 \) \textbf{as} \( \|\delta F\| \rightarrow 0 \) \textbf{(i.e. identify the structurally stable front)}
\end{enumerate}

Technically require RG because \( \delta F \) gives secular divergences in perturbation theory.
**RG(2)**

**Example 1**: Marginal Perturbation

\[ \partial_t \phi = \partial_x^2 \phi + \phi(1-\phi) + \epsilon \phi(1-\phi) \]

\[ \text{RG} \Rightarrow c = 2 + \epsilon + O(\epsilon^2) \]

\[ \text{Exact} \Rightarrow c = 2 \sqrt{1 + \epsilon} \]

**Example 2**: Selection

**Detailed Calculation**

\[ \delta c = \sqrt{\Delta} \quad c = 2 \]

\[ \delta c = \sqrt{c^2 - 4} \quad c > 2 \]

As \( \Delta \to 0 \)

Conclude that \( c = 2 \) is only structurally stable front.
P-SMALL PERTURBATIONS (2)

CONCLUDE: ONLY \( \hat{\omega} \)ORIG\( \text{INAL} \) IS STRUCTURALLY

STALL. OUR HYPOTHESIS IMPLIES THAT

THIS VALUE SHOULD BE THE UNIQUE

OBSEENRABLE SPEED. (CORRECT RESULT)

[Technically we can prove that minimum speed of

Fisher's eqn is structurally stable. Aronson &

Weinberger proved min speed is selected.]

OTHER TESTS OF STRUCTURAL STABILITY HYPOTHESIS:

\( \omega \geq \omega_\text{G} - \delta_\text{G} \omega + \frac{\omega}{\delta} (1+\omega)(1-\omega), \quad \delta \leq 1/12 \)

- Solve numerically
- \( 8\omega \) destroys all but
  one one front
- Converges as \( \Delta \to 0 \)
  to selected wave
- \( 0 < b < 1/2 \) "Marginal stability" fails but S\&H
  gives correct result
Cell Dynamical Systems
PHASE DIAGRAM OF A-B ALLOY

Critical Point

TWO PHASE COEXISTENCE

Coexistence Curve

Spinodal Line

Order Parameter

(Concentration of A atoms)

$\frac{df}{dc} = 0$

$\frac{df}{dc} = 0$

$f(c, T_1)\text{ coarse-grained free energy density}$

$\frac{df}{dc} > 0$

$\frac{df}{dc} < 0$

thermodynamic free energy density

$C_1 \quad C_2$
Scaling in Spinodal Decomposition

Statistically indistinguishable

Data collapse / scaling
CELL DYNAMIC SIMULATION OF SPINODAL DECOMPOSITION (Oono + Puri)
ISOSURFACES WHERE

$C_A = C_B$

Figure V.1: $\psi = 0$ isosurfaces of 3-dimensional binary alloy spinodal decomposition at critical quench. Parameters are $M_{oa} = 1$, $D_{oa} = 0.7$, and $A = 1.15$ for tank map described in text. System size is $192^3$. The images are a $128^3$ subset of the entire sample. Images are, on the left from top to bottom, of $t_{oa} = 1000, 2000, 4000$ data and on the right from top to bottom, of $t_{oa} = 7000, 12000$ and $20000$ data.
Time sequence showing the coarsening of microstructure during spinodal decomposition of a binary alloy, such as those used for high strength materials (Li-Al) or porous networks (Vicor glass).

- Cell Dynamic System (CDS) algorithm is a thousand times faster than standard Monte Carlo methods.

- Realistic simulations of microstructure resulting from different quenches and materials parameters are now feasible with this method.

- *Used by other researchers world-wide, this algorithm was invented and developed at the Illinois MRL.*
Kinetics of Phase Separation in Mn_{0.67}Cu_{0.33}

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and
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(Received 9 March 1987)

We have examined the kinetics of phase separation in Mn_{0.67}Cu_{0.33} using time-resolved neutron-scattering techniques. In an early-time regime, the kinetics follows the Cahn-Hilliard-Cook linear theory of spinodal decomposition. There is an intermediate stage, then, as a late time, dynamic scaling is obeyed. The time dependence of the wave vector at maximum scattering intensity (which is inversely proportional to the average linear domain size) can be well described over the entire late-time regime and much of the intermediate-time regime by arguments recently put forward for earlier-time corrections to the limiting late-time stages of phase separation.

PACS numbers: 64.75 +g, 61.10.Ks, 81.40.Cd

![Early Time Diagram](image)

**Fig. 1.** Top: Scaling behavior of the data sets of \( F/Q_{\text{max}} = Q_{\text{max}} S(Q) \) for times exceeding 5000 sec. Bottom: Same data sets as the top panel, as well as five data sets at 965, 1602, 2299, 2886, and 3523 sec. These earlier-time data sets depart from the scaling relation in a systematically greater extent with decreasing time.

![Late Time Diagram](image)

**Fig. 2.** The \( S(Q,t) \) data sets at relatively early times. The solid lines are the results of the fit of the CHT theory with the \( S(Q) \) base function shown as the bottom of the figure to the data. Systematic discrepancies are clear for times longer than 300 sec. Data sets have zero \( S(Q) \) values shown by the solid lines to the right of the panels for clarity of presentation.
**Simple Linear Theory**

Cahn-Hilliard "Coarse-Grained" Free Energy

\[ F\{c(x,t)\} = \int d^d x \left[ \frac{1}{2} (\nabla c)^2 + f(c(x,t)) \right] \]

- **Coarse-Grained** on scale \( \Lambda \)
- I.e. no \( k \)-modes with \( k > \Lambda \)
- **Free Energy Cost** to maintain concentration gradient

**Mass Conservation**

\[ \partial_t c + \nabla \cdot \mathbf{I} = 0 \]

**Phenomenology (Linear Response)**

\[ \mathbf{I} = - \mathbf{M} \nabla \frac{\delta F}{\delta c} \]

- **Mobility**

**Growth of Small Perturbations**

\[ c(x) - c_0 \sim e^{ikx + \omega(k)t} \]

\[ W(k) \]

- **Unstable**
- **Stable**
DYNAMIC SCALING

\[ S(k,t) = \int d^d \xi \, e^{i k \cdot \xi} \langle \psi(\xi,t) \psi(0,t) \rangle \]

SCATTERING FUNCTION

IN GENERAL \( \psi \) IS ORDER PARAMETER

EXPERIMENT & NUMERICS GIVE

\[ S(k,t) \xrightarrow{t \to \infty} L(t)^d F(kL(t)) \]

\[ L(t) \sim t^\phi \]

IN SPINODAL DECOMPOSITION \( \phi = 1/3 \)

\[ L(t) \] IS CHARACTERISTIC SCALE OF PATTERN i.e. \( \exists \) only one such scale
COMPUTATIONAL TECHNIQUE
CONVENTIONAL

MODEL AS P.D.E.
DISCRETIZE P.D.E.
COUPLED MAPS
COMPUTE

CDS

MODEL AS COUPLED MAPS
COMPUTE

"It was bound to happen—
they're beginning to think like binary computers."
CDS PHILOSOPHY:

- Identify key physics
- Model this faithfully
- Remaining ingredients added after

EXAMPLES:

<table>
<thead>
<tr>
<th>Alloy Phase Separation</th>
<th>Hydrodynamics and Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>On-site &quot;double well&quot; potential + diffusion + conservation</td>
<td>Advection + viscosity + driving force</td>
</tr>
</tbody>
</table>
Prototype CDS: Spinodal Decomposition

Y. Oono + S. Puri, Phys. Rev. Lett. 58, 836 (87)
Phys. Rev. A 39, 434 (89); ibid. 39, 1542 (89).

Coarse-grain system on scale of correlation length

In each cell, concentration relaxes in "potential"
c from free energy dynamics has three fixed points

- Model cell dynamics by any map which has fixed point structure of evolution expected on physical grounds, not discretisation of PDE!

- Couple cells: cell concentration also evolves according to the average concentration of neighbouring cells.

- Implement conservation law
LEVEL OF DESCRIPTION

SCALAR
DEFECTS ARE
DOMAIN WALLS

VECTOR
DEFECTS ARE VORTICES

L(t)
CHARACTERISTIC
SCALE OF PATTERN

CORRELATION LENGTH

ORDER PARAMETER

MICROSCOPIC
DEGREES OF FREEDOM

COARSE-GRAINED
DEGREES OF FREEDOM (\gamma)

INTERESTED IN PHENOMENA ON SCALE
OF L(t). IDEALLY COULD IGNORE
SCALES < \xi.

THERMAL FLUCTS. \rightarrow SCALES < \xi.

\frac{\xi}{L(t)} \rightarrow 0 \text{ as } t \rightarrow \infty, \text{ so ignore}
THERMAL FLUCTUATIONS (\Gamma = 0)
KEY POINTS:

(1) SPACE + TIME DISCRETE.
    \Delta x \sim \Delta t \sim O(1)
    NO CONTINUUM LIMIT

(2) POTENTIAL: ONLY FIXED
    POINT STRUCTURE IMPORTANT
(1) MORE IMPORTANT THAN (2).

ASSUMPTION:

THIS ALGORITHM IS IN UNIVERSALITY
CLASS OF P.D.E. NOT OBVIOUS!

NOTE: IN LITERATURE, CDS APPROACH
USED WITH P.D.E., FOLLOWING (1) BUT
NOT (2) i.e. DOING EULER BUT NOT
CONTINUUM LIMIT.
CDS FOR SPINODAL DECOMPOSITION

\[ C_t(n) = \text{CONCENTRATION (i.e. OF SPECIES A) IN CELL AT } n \text{ AT TIME STEP } t. \]

EQUATION OF MOTION:

\[ C_{t+1}(n) = C_t(n) + I_t(n) - \langle I_t(n) \rangle \]

EVOLUTION:

\[ F(x) = A \tanh \left( \tanh^{-1} \left( \frac{A}{x} \right) \right), \quad A = 1.15 \]

\[ I_t(n) = F(C_t(n)) - C_t(n) + D(\langle C_t(n) \rangle - C_t(n)) \]

SINGLE CELL DYNAMICS

COUPLING BETWEEN CELLS = DIFFUSION

\[ D = 0.7 \]

AVERAGING:

\[ \langle U \rangle = \frac{1}{6} \sum_{j \text{ nearest}} U_j + \frac{1}{12} \sum_{j \text{ next nearest}} U_j \]

NEAREST NEIGHBOURS

NEXT NEAREST NEIGHBOURS
SUMMARY OF TIME REGIMES

O.P. FLUCTUATIONS OF INFINITESIMAL AMPLITUDE, LONG WAVELENGTH GROW EXPONENTIALLY (LINEAR THEORY) (NON-UNIVERSAL)

\[ \downarrow \]

NONLINEARITY → STABILISATION → PATTERN WITH DOMAIN WALLS

\[ \downarrow \]

DOMAIN WALL DYNAMICS GOVERNED BY CURVATURE → SCALING REGIME

(NONLINEAR THEORY, RECENT PROGRESS) (UNIVERSAL)

\[ \downarrow \]

EQUILIBRATION AS EFFECTS OF BOUNDARIES ARE FELT. (NON-UNIVERSAL)

\[ t \]

INTERMEDIATE ASYMPTOTIC UNIVERSAL SCALING REGIME
VORTEX ANNihilation IN O(2) MODEL

\[ \Psi(c,0) = 0 \]

ONE SCALE
DEFECTS

Preliminary CDS simulation of isotropic-nematic transition

Schlieren patterns show evolution of structure and defect annihilation

100 x 100 lattice

$t = 500, 1000, 2000$

Tensor order parameter
Fig. 2. Time evolution of the disclination texture.
Fig. 6. The scaled correlation function of defect densities. The solid line and the large closed squares indicate the theoretical curve by Lue et al.\textsuperscript{11)} and the data of simulation by Mondello et al.\textsuperscript{8)} respectively.
Block Spins for Partial Differential Eqns.


**PERFECT OPERATORS: MOTIVATION**

**GOAL:** Seek numerical methods where grid (and time step) size is determined by the physics of the solution, not the accuracy of the approximation method.

**Typical behaviour of conventional numerics**

**Desired behaviour of numerical scheme**
**Perfect Operators**

\[ f \rightarrow f' \]

**Sampling**

\[ \text{Sampling/Coarse-Graining give Continuum Limit} \]

\[ dx \rightarrow 0 \]

**Recursion Relation between Operators at Different Levels of Resolution**

\[ dx \rightarrow 0, \quad a = \text{constant} \]

**Fixed Point Operator: Perfect Representation at Scale \( a \)!!**
Perfect Operators: Basic Idea.

Time Evolution via
FDE

\[ u(x,0) \xrightarrow{\text{Ni, Ca}} u(x,t) \]

COARSE GRANING OPERATOR
FROM CONTINUOUS TO SCALE CL.

\[ u(x,0) \xrightarrow{\text{Ca}} U(x,t) \]

Example: \[ \partial_t u = \partial_x^2 u + \xi \]

DIFFUSION NOISE

\[ e^{-\frac{1}{2} \sum \Delta_u \Delta_x u} = \int Du e^{-\frac{1}{2} \int \partial_x^2 u} \prod \delta (U(a) - \int_{a/2}^{a/2} \partial_x^2 u(x + na)) \]

FIELD ON LARGE SCALE
LAPLACIAN ON LATTICE OF SPACING CL.
LABELS LATTICE POINTS
INTEGRATE OUT SMALL SCALE DEGREES OF FREEDOM

\[ \Delta_u U(a) = -\frac{1}{a^2} \sum_{\mathbf{R}} U(a(x+\mathbf{R})) \rho(\mathbf{R}) \]

\[ \rho(\mathbf{R}) = \int \left\{ e^{-i\mathbf{p}.\mathbf{R}} \right\} f(\mathbf{p}) \]

\[ f(\mathbf{p})^{-1} = \sum_{n=0}^{\infty} \frac{1}{(\mathbf{p} + 2\pi n)^2} \frac{i}{C^{2}} + \frac{1}{3} \]

\[ d=1 \rho(\mathbf{R}) = -6\frac{K}{K-1} \left( \frac{3K}{K+4} \right)^{\frac{K}{2}} \left[ \frac{(3K(K+4))^{\frac{K}{2}} - 2(1+K)}{K-2} \right]^{1/2} , \mathbf{R} \neq 0 \]
RANDOM INITIAL CONDITIONS

\[ dx = 0.5 \quad N = 1024 \]

Inset: Power spectrum
COMPARISON OF RESULTS FOR THE
DIFFUSION EQUATION
\[ \partial_{t} u = \partial_{x}^{2} u \]

\[ t = 100 \quad N = 128 \]

BENCHMARK \( dx = 0.5 \quad N = 1024 \quad dt = 0.001 \)
RG \( dx = 4 \quad dt = 5 \)

UNIFORM

---

Diagram:

- bench mark
- coarse grained
- uniform sample
SPACE-TIME STOCHASTIC DYNAMICS

IN GENERAL IT IS NOT CORRECT TO COARSE GRAIN IN SPACE ONLY. RG ITERATIONS INVOLVE $\partial_\epsilon$ OPERATOR. THIS GENERATES NON-TRIVIAL TIME DEPENDENCE ON LARGE SCALES

SOLUTION: COARSE-GRAIN ENTIRE EQUATION IN SPACE-TIME.

METHOD: MARTIN-SIGGIA-ROSE (MSR) PATH INTEGRAL

$$\partial_\epsilon \psi = - f(x,t;\psi) + \eta(x) \quad \text{--- NOISE}$$

$$\Rightarrow P(\psi, t; \psi_0, t_0) = \int D\phi D\eta e^{-\frac{\Delta x \Delta t}{2\hbar} \sum_{i=1}^{N} \left[ \frac{\partial^2}{\partial \phi_i^2} - \frac{\hbar}{2 \Delta x} \partial_i \eta \right] + \frac{1}{8} \left( \eta - \partial_\phi \phi - f(x) \right)^2}$$

$(\Delta x, \Delta t)$ DISCRETISATION IN $(x,t)$

$P(\psi, t; \psi_0, t_0)$ CONDITIONAL DISTRIBUTION
RESULTS FOR LINEAR PROBLEM

\[ \partial_t \varphi = \Delta_x \varphi - m \varphi + \eta \]

PERFECT ACTION:

\[ P[\varphi_t, t; \varphi_0] \sim \int D\varphi D\eta e^{-\int \varphi H^{-1} \varphi} \]

\[ H^{-1}(k_\nu) = -\partial_t G - r \sin \frac{\omega \nu}{2} \delta_y G + 2 r \sin \frac{\omega \nu}{2} \times \]

\[ \sum_i \frac{4 \sin^2(k_i \nu)}{k_i^2 (k_i^2 + \nu)^3} \cdot e^{-\frac{\omega \nu}{\cosh \omega \nu - \cos \omega \nu}} + \frac{1}{3 \nu \omega} \]

WHERE

\[ \omega_i = \frac{(k_i^2 + \nu)}{r} \]

\[ G(k, \nu) = \frac{\sum \sin^2(k_i \nu)}{(k_i^2 + \nu)} = \frac{1}{\mu} \left[ 1 - \sinh \sqrt{\nu} (1 - \cosh \omega) \right] \]

\[ \mu = \max \Delta x^2 \]

\[ r = \Delta x^2 / \Delta t \]

\[ \kappa = \text{arbitrary parameter (tuned to minimize spatial extent)} \]

\[ d = 1 \text{ if } \partial^d x \approx \text{even} \]

\[ 0 \text{ if } \partial^d \varphi \]
FIG. 1. RG flow of the dynamics operator. We start from a microscopic form of the Laplacian operator $L_{0, \omega} = i\omega + \frac{1}{\epsilon^2} k^2$. The functional form of $L_\omega$ is, $(L_{\omega,n})^{-1} = \frac{1}{2\pi} \frac{\omega_\beta + \frac{\alpha_n}{\Delta = \pi}}{f_n(k)}$. 
FIG. 2. Surface plot of amplitude of perfect action operator coefficients for the diffusion equation. The coefficients exponentially decay away from the origin. The decay speed is slow along the x direction. $\kappa = 2$, $m = 0$ and $\Delta t = \Delta x^2$. 
**Dispersive Relation**

As very well with exact result!

![Graph](image)

**FIG. 5.** Decay rate of wave modes for diffusion equation. \( \kappa = 2, \mu = 0.25 \) and \( \Delta x^2/\Delta t = 1 \). Perfect operator decay rate obtained using the first two \( t \neq 0 \) nodes (equation 32).

*NA means usual "numerical analysis" discretisation (central differences)*

\* \( S(k,t) \) calculated \( \sim e^{-\omega(k)t} \rightarrow \omega(k) \)
FIG. 6. Cross sections of $S(k, \omega)$ for the diffusion equation. $m = 1, L = 16, T = 8$. Cross sections are at $\omega = 0$ (left) and $k = 0$ (right). The exact result is $\left\{\left[(m + k^2)^2 + \omega^2\right]\right\}^{-1}$. 
COMPUTATIONAL COMPLEXITY

PERFECT ACTION NON-LOCAL.

⇒ AT EACH GRID \((x,t)\) POINT NEED 105 POINTS EVALUATED.

ONLY 7 EVALUATION CONVENTIONALLY.

32 x 32 LATTICE 10,000 MC STEPS

\[ \text{NA} : 30.1 \text{ s} \]
\[ \text{PO} : 128 \text{ s} \]

BUT VARIANCE IS REDUCED BY PERFECT OPERATORS

ESTIMATE THAT EVEN WITH HIGHER OVERHEAD,
TO REACH COMPARABLE ACCURACY, CONVENTIONAL
TECHNIQUES REQUIRE \(\approx 40\) TIMES AS
MANY FLOATING-POINT OPERATIONS.

(IN 1 + 1 DIMENSIONS)
\[ \partial_t \varphi = \partial_x^2 \varphi - m \varphi + g \varphi^3 + \eta \]

Write action \( S = S_0 + S_1 \)

\[ S_0 = \varphi (-\partial_x^2 + (-\partial_x^2 + m)^2) \varphi \]
\[ S_1 = 2g \varphi^3 (-\partial_x^2 + m) \varphi + (g \varphi^3)^2 - \frac{3g \varphi^2}{\Delta x} \varphi^2 \]

Problem: Generate many non terms under RG iteration

Heuristic solution: Use perfect form for \( S_0 \) and use \( S_1 \) as originally written. Reasonably effective.
\[ \partial_t \varphi = \partial_x^2 \varphi - m \varphi + g \varphi^3 + \eta \]

\[ n > 0 \]

FIG. 10. Decay rates of wave modes for the Ginzburg-Landau equation. \( m = 1, L = 16, T = 8 \).
\[ \dot{\psi} = \partial_x^2 \psi - m \psi + g \psi^3 + \eta \]

\[ m < 0 \]

FIG. 13. Decay rates of wave modes for the Ginzburg-Landau equation. \( m = -1, L = 16, T = 8 \).
Conclusions

1. "Exact" numerical results can be obtained with minimal effort by modelling the right level of description.

2. RG methods computationally efficient in problems with scale invariance or weak scale interference.

3. Stochastic evolution equations & sub-grid models can be approached by RG. Good results but hard.

Next steps: Kraichnan model for passive scalars

Navier-Stokes (t & x space)
SUMMARY

1. RG METHODS PROVIDE A UNIFIED WAY TO APPROACH PERTURBATION AND SINGULAR PERTURBATION PROBLEMS

2. NO ASYMPTOTIC MATCHING REQUIRED

3. NO A PRIORI GUESSWORK IN WRITING-DOWN FORM OF PERTURBATION EXPANSION

4. RESULTS UNIFORMLY VALID AND PRACTICALLY ACCURATE, EVEN FOR $\varepsilon \sim O(1)$

5. SYSTEMATIC WAY TO OBTAIN REDUCTIVE DYNAMICS

6. RESULTS NEED EXTENSIVE MATHEMATICAL VALIDATION, ELUCIDATION AND RIGORISATION
End of lecture 3