RENORMALISATION GROUP AND THE ASYMPTOTICS OF PARTIAL DIFFERENTIAL EQUATIONS

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The Renormalization Group in Applied Mathematics

Professor Nigel Goldenfeld

Lecture 1: Self-similarity, incomplete similarity and asymptotics of nonlinear PDEs

Wednesday May 21, 2003 at 12.00 noon.

Dimensional analysis; extended dimensional analysis and anomalous exponents in the long-time behaviour of PDEs; modified porous medium equation; propagation of turbulence.

Lecture 2: Singular perturbations: uniformly valid approximations from RG

Wednesday May 28, 2003 at 12.00 noon.

Perturbed oscillators, boundary layer problems with log ε terms, WKB with turning points, switchback problems; spatially-extended systems and the derivation of amplitude and phase equations near and far from bifurcations.

Lecture 3: Numerical methods and under-resolved computation

Friday May 30, 2003 at 12.00 noon.

Similarity solutions are fixed points of RG transformations; velocity selection, structural stability and the Kolmogorov-Petrovsky-Piscunov problem; universal scaling phenomena in stochastic PDEs; perfect operators.

R.P. FEYNMAN

Lectures in Physics Vol 2

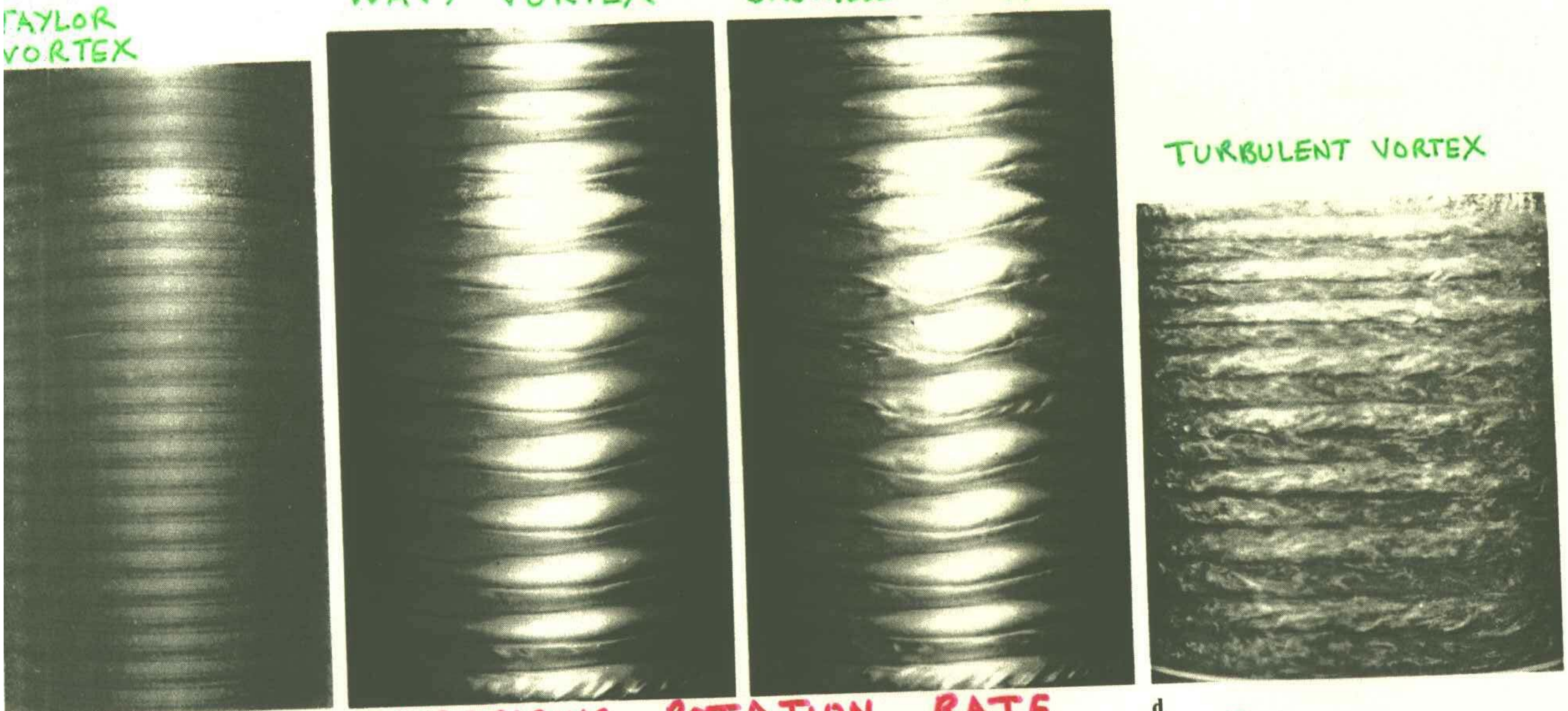
We have written the equations of water flow. From experiment, we find a set of concepts and approximations to use to discuss the solution—vortex streets, turbulent wakes, boundary layers. When we have similar equations in a less familiar situation, and one for which we cannot yet experiment, we try to solve the equations in a primitive, halting, and confused way to try to determine what new qualitative features may come out, or what new qualitative forms are a consequence of the equations. Our equations for the sun, for example, as a ball of hydrogen gas, describe a sun without sunspots, without the rice-grain structure of the surface, without prominences, without coronas. Yet, all of these are really in the equations; we just haven't found the way to get them out.

There are those who are going to be disappointed when no life is found on other planets. Not I—I want to be reminded and delighted and surprised once again, through interplanetary exploration, with the infinite variety and novelty of phenomena that can be generated from such simple principles. The test of science is its ability to predict. Had you never visited the earth, could you predict the thunderstorms, the volcanos, the ocean waves, the auroras, and the colorful sunset? A salutary lesson it will be when we learn of all that goes on on each of those dead planets—those eight or ten balls, each agglomerated from the same dust cloud and each obeying exactly the same laws of physics.

The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality—or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

(a) E. L. KOSCHMIEDER (1979)
(b) D. COLES (1965)
(c) D. COLES (1965)
(d) P. FENSTERMACHER et al (1979)

FLOW BETWEEN CONCENTRIC CYLINDERS WITH INNER CYLINDER ROTATING



a b c d

Fig. 6.1a–d. Photographs of the flow between concentric cylinders with the inner cylinder rotating. (The radius ratio is 0.88.) **(a)** $R \simeq R_c$; Taylor vortex flow [6.5]. **(b)** $R/R_c = 10.4$; wavy vortex flow [Ref. 6.6, Fig. 19d]. **(c)** $R/R_c = 12.3$; the "first appearance of randomness" in wavy vortex flow [Ref. 6.6, Fig. 19e]. **(d)** $R/R_c = 23.5$; the azimuthal waves have disappeared and the flow is turbulent, although the axial periodicity remains [Ref. 6.7, Fig. 1d]. The visualization of the flow in these experiments was achieved by suspending small flat flakes in the fluid; the flakes align with the flow, and variations in their orientation are observed as variations in the transmitted or reflected intensity

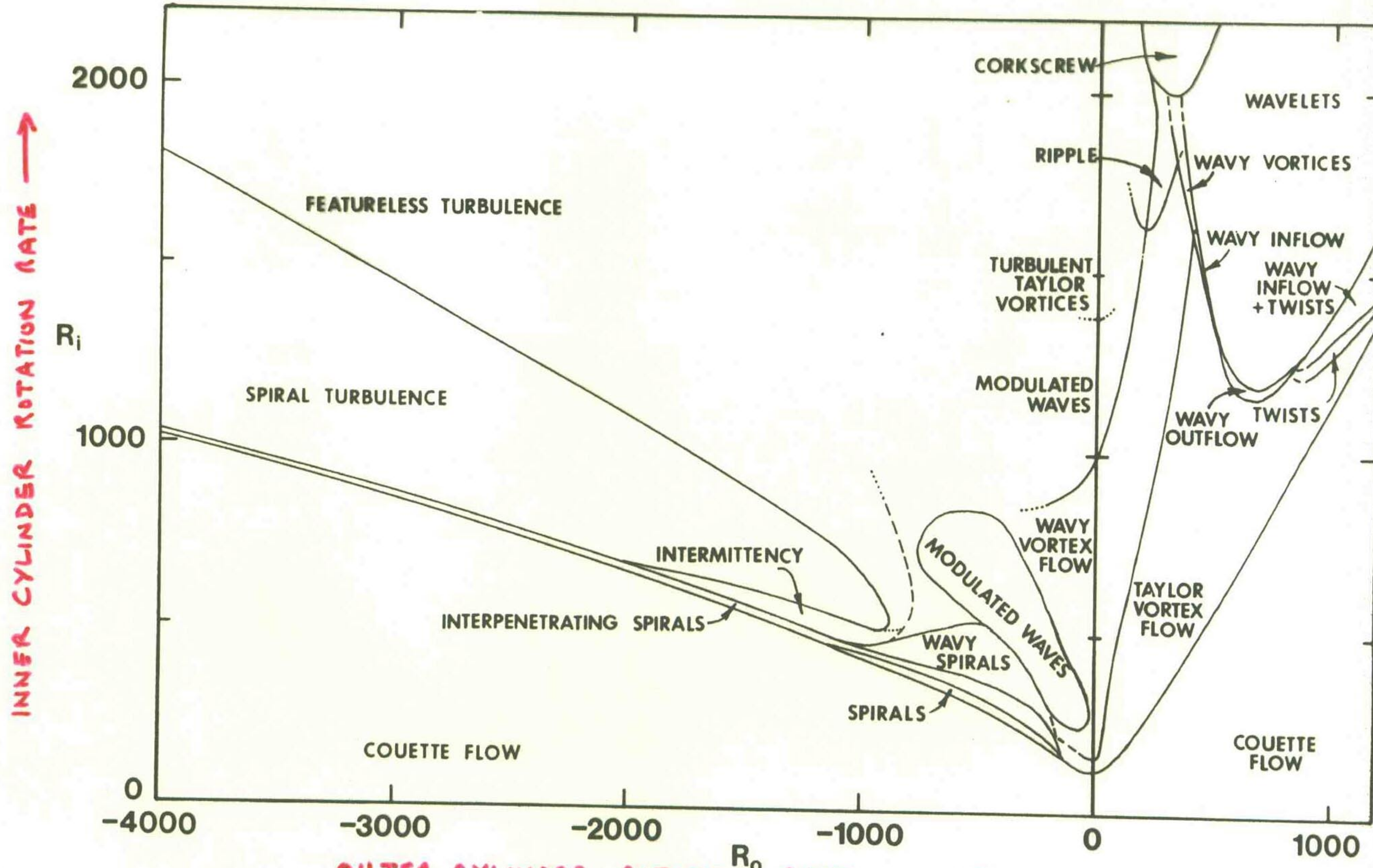
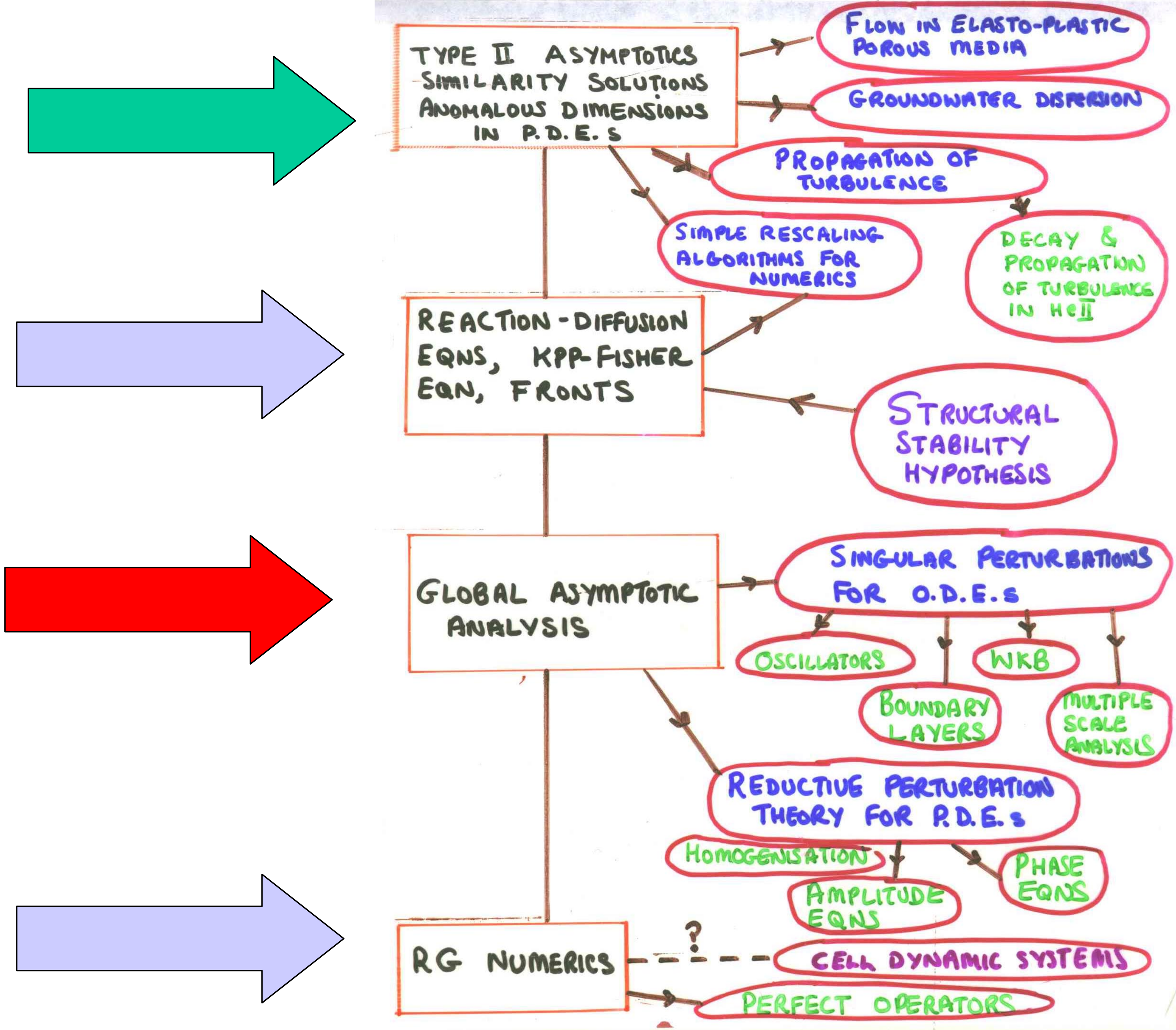


Fig. 10.1. Flow regimes observed in flow between independently rotating cylinders with radius ratio 0.883 and aspect ratio 30. (Adapted from [10.40])

C. D. ANDERECK et al (1985).

N.B. PRECISE STATE IS NOT A UNIQUE FUNCTION OF R_i, R_o, R_i/R_o and ASPECT RATIO


TYPE II ASYMPTOTICS SIMILARITY SOLUTIONS ANOMALOUS DIMENSIONS IN P.D.E.s
FLOW IN ELASTO-PLASTIC POROUS MEDIA
GROUNDWATER DISPERSION
PROPAGATION OF TURBULENCE
SIMPLE RESCALING ALGORITHMS FOR NUMERICS
DECAY & PROPAGATION OF TURBULENCE IN He II
REACTION-DIFFUSION EQNS, KPP-FISHER EQN, FRONTS
STRUCTURAL STABILITY HYPOTHESIS
GLOBAL ASYMPTOTIC ANALYSIS
SINGULAR PERTURBATIONS FOR O.D.E.s
OSCILLATORS
WKB
BOUNDARY LAYERS
MULTIPLE SCALE ANALYSIS
REDUCTIVE PERTURBATION THEORY FOR P.D.E.s
HOMOGENISATION
AMPLITUDE EQNS
PHASE EQNS
RG NUMERICS
?
CELL DYNAMIC SYSTEMS
PERFECT OPERATORS

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   INTERMEDIATE ASYMPTOTICS OF 2nd KIND

   HEURISTIC RG CALCULATION

   PERTURBATION THEORY

   GEOMETRICAL INTERPRETATION

2. GLOBAL ASYMPTOTICS

   MULTISCALE PROBLEMS

   BOUNDARY LAYERS

   WKB

   SWITCHBACK PROBLEMS

3. REDUCTIVE PERTURBATION THEORY

Anomalous dimensions

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N. D. Goldenfeld, O. Martin and Y. Oono. Intermediate asymptotics and renormalization group theory. J. Scientific Computing *4*, 355-372 (1989).

N. D. Goldenfeld, O. Martin, Y. Oono and F. Liu. Anomalous dimensions and the renormalization group in a nonlinear diffusion process. Phys. Rev. Lett. *64*, 1361-1364 (1990).

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All my RG papers can be obtained in reprint form from

http://guava.physics.uiuc.edu/~nigel/articles/RG

Similarity, Self-Similarity, and Intermediate Asymptotics

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LECTURES ON PHASE TRANSITIONS AND THE RENORMALIZATION GROUP

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see ch. 10 especially

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The Advanced Book Program

Reading, Massachusetts Menlo Park, California New York
Don Mills, Ontario Wokingham, England Amsterdam Bonn
Sydney Singapore Tokyo Madrid San Juan
Paris Seoul Milan Mexico City Taipei

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Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations, Comm. Pure Appl. Math. **47**, 893-922 (1994)

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Stability of Moving Fronts in the Ginzburg-Landau Equation, Commun. Math. Phys. **159**, 287-318 (1994)

Global large time self-similarity of a thermal-diffusive combustion system with critical nonlinearity, J. Diff Eqn, Vol. 130, No. 1, 1996, pp 9-35

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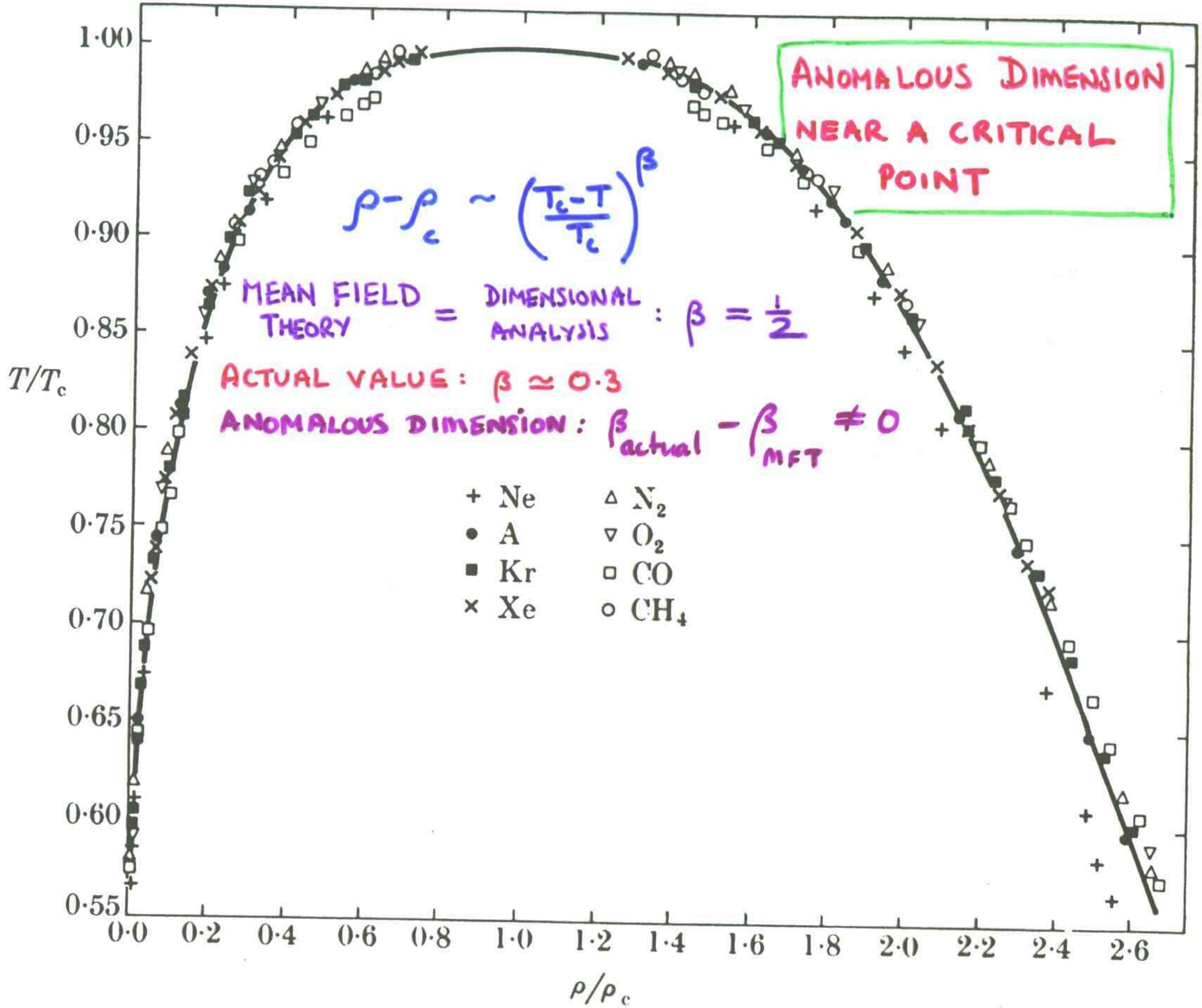


FIG. 1.8. Measurements on eight fluids of the coexistence curve (a reflection of the $P\rho T$ surface in the ρT plane analogous to Fig. 1.3). The solid curve corresponds to a fit to a cubic equation, i.e. to the choice $\beta = \frac{1}{3}$, where $\rho - \rho_c \sim (-\epsilon)^{\beta}$. From Guggenheim (1945).

SIMILARITY SOLUTIONS

IN NON-EQUILIBRIUM PROBLEMS, WE ARE OFTEN INTERESTED IN SIMILARITY SOLUTIONS

$$\boxed{u(x,t) = t^{\alpha} f(x t^{\beta})}$$

OR TRAVELLING WAVES

$$\boxed{u(x,t) = f(x - vt)}$$

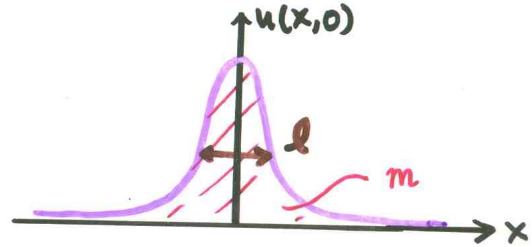
REASON: THESE SOLUTIONS OFTEN DESCRIBE LONG TIME BEHAVIOUR

GOAL: COMPUTE EXPONENTS α, β
VELOCITY v
SCALING FUNCTION f

SUFFICES TO CONSIDER SIMILARITY SOLUTIONS ONLY: SUBSTITUTION $x = \log X$ $t = \log T$ CONVERTS

$$\underset{\text{TRAVELLING WAVE}}{f(x-vt)} \longrightarrow \underset{\text{SIMILARITY SOLUTION}}{F\left(\frac{X}{T^{v}}\right)}$$

DIFFUSION EQUATION



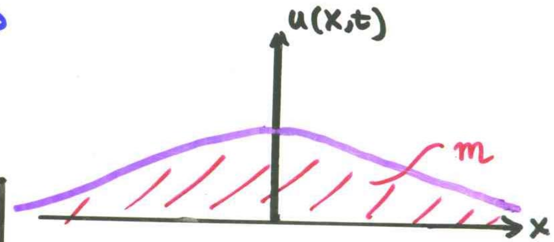
INITIAL VALUE PROBLEM:

$$\boxed{\partial_t u = \frac{1}{2}\partial_x^2 u}$$

GAUSSIAN OF WIDTH l AND MASS m

$\downarrow t$

$$\boxed{u(x,t) = \frac{m e^{-x^2/2(t+l^2)}}{\sqrt{2\pi(t+l^2)}}}$$



LONG TIME BEHAVIOUR:

$$u(x,t) \xrightarrow[l\text{ fixed}]{t\to\infty} \frac{m e^{-x^2/2t}}{\sqrt{2\pi t}}$$

OR EQUIVALENTLY

$$u(x,t) \xrightarrow[t\text{ fixed}]{l\to 0} \frac{m e^{-x^2/2t}}{\sqrt{2\pi t}}$$

I.e. ASYMPTOTIC BEHAVIOUR OF INITIAL VALUE PROBLEM GIVEN BY SIMILARITY SOLUTION — THE SOLUTION CORRESPONDING TO DELTA FUNCTION INITIAL CONDITION.

DIMENSIONAL ANALYSIS

DIFFUSION EQUATION EXAMPLE OF COMMON PHENOMENON IN PHYSICS.

EXPRESS PHYSICAL PROBLEM IN DIMENSIONLESS VARIABLES $\Pi, \Pi_0, \Pi_1, \Pi_2, \ldots \Pi_n$

THEN SOLUTION IS OF FORM

$$\boxed{\Pi = f(\Pi_0, \Pi_1, \Pi_2, \cdots, \Pi_n)}$$

IF ONE VARIABLE (e.g.) Π_0 IS SMALL, THEN USUALLY SET $\Pi_0 = 0$.

i.e.

$$\Pi_0 = \frac{\text{characteristic dimension of apparatus}}{\text{radius of moon}} \approx 0$$

THEN WE HAVE

$$\boxed{\Pi = f(0, \Pi_1, \Pi_2, \ldots, \Pi_n)}$$

"COMMON SENSE" = CASE 1

IN DIFFUSION EQUATION EXAMPLE

$$\Pi = \frac{u}{m}\sqrt{t} \;;\quad \Pi_0 = \frac{\ell}{\sqrt{t}} \;;\quad \Pi_1 = \frac{x}{\sqrt{t}}$$

$$\boxed{u = \frac{m}{\sqrt{t}}\, f\left(\frac{x}{\sqrt{t}}\right) \quad \text{as } \Pi_0 \to 0}$$

DIMENSIONAL ANALYSIS (2)

WE MADE A STRONG ASSUMPTION THAT THE LIMIT $\Pi_0 \to 0$ EXISTS. BARENBLATT HAS GIVEN SEVERAL EXAMPLES WHERE THIS ASSUMPTION BREAKS DOWN.

CLASSIFY ASYMPTOTICS:

CASE 1: $\boxed{\Pi \sim f(0, \Pi_1, .. \Pi_n) \quad \text{as } \Pi_0 \to 0}$

COMMONPLACE (BY CONSTRUCTION)

CASE 2: $\boxed{\Pi \sim \Pi_0^{-\alpha} g\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, \cdots \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right) \quad \text{as } \Pi_0 \to 0}$

PRESENTS PROBLEMS WHEN IT OCCURS. FUNCTION g AND THE EXPONENTS $\alpha, \alpha_1, \ldots \alpha_n$ MUST BE DETERMINED.

CASE 3: NONE OF THE ABOVE

CASE 2 EXAMPLES IN FLUID MECHANICS, CRITICAL PHENOMENA, ELECTROMAGNETISM,......

THESE PROBLEMS CAN BE ANALYSED USING THE RENORMALISATION GROUP.

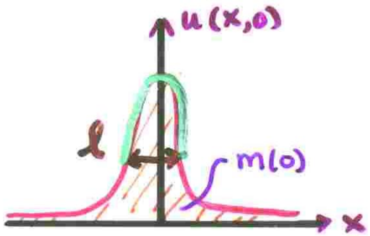
BARENBLATT EQUATION

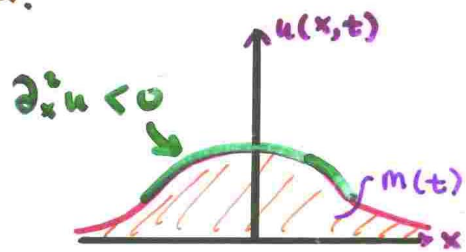
SEEMINGLY INNOCUOUS MODIFICATION TO DIFFUSION EQN.

$$\boxed{\partial_t u = D\,\partial_x^2 u \qquad D = \begin{cases} \frac{1}{2} & \partial_x^2 u > 0 \\ \frac{1}{2}(1+\epsilon) & \partial_x^2 u < 0 \end{cases}} \quad \text{(B)}$$

DESCRIBES PRESSURE IN A FLUID PASSING THROUGH A POROUS MEDIUM WHICH CAN EXPAND AND CONTRACT IRREVERSIBLY (P2C2E).

PARAMETER ϵ DEPENDS UPON ELASTIC CONSTANTS OF FLUID, POROUS MEDIUM.





(B) IS NOT DERIVABLE FROM CONTINUITY EQN

$$\partial_t u + \nabla \cdot \underline{j} = 0$$

SO MASS OF DISTRIBUTION NOT CONSERVED:

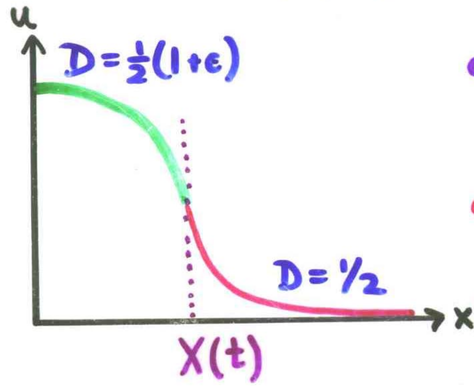
$$\boxed{m(t) \neq m(0)}$$

BARENBLATT EQN (2)

Q/ WHAT IS LONG TIME BEHAVIOUR OF (B) ?

A/ ¿ $u(x,t) \xrightarrow[t\to\infty]{} \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}, \epsilon\right)$?

NO !



- SUBSTITUTE PROPOSED FORM INTO (B).
- GIVES TWO O.D.E.'s
- CANNOT MATCH 1st + 2nd DERIVATIVES AT X(t)

BUT THERE EXISTS A UNIQUE SOLUTION OF THE INITIAL VALUE PROBLEM WITH CONTINUOUS SECOND DERIVATIVES (KAMENOMOSTSKAYA, 1957)

WE WILL SEE THAT LONG TIME BEHAVIOUR IS

$$\boxed{u(x,t) \xrightarrow[t\to\infty]{} \frac{1}{t^{\frac{1}{2}+\alpha}} f\left(\frac{x}{\sqrt{t}}, \epsilon\right)}$$

anomalous dimension, $\alpha = \alpha(\epsilon)$

HEURISTIC DERIVATION

WRITE SOLUTION AS

$$\boxed{u(x,t) = \frac{m(t)}{\sqrt{2\pi(t+\ell^2)}}\, e^{-x^2/2(t+\ell^2)}}$$

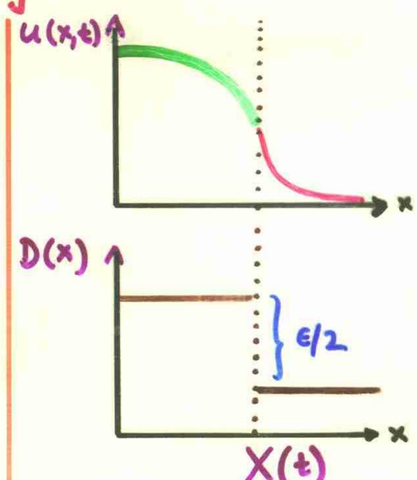
IF ϵ IS SMALL, REMOVAL OF MASS OCCURS "SLOWLY" AND DISTRIBUTION ADIABATICALLY ADJUSTS TO THE GAUSSIAN FORM ABOVE (STRICTLY VALID FOR $\epsilon = 0$).

EQUATION OF MOTION FOR $m(t)$:

$$\partial_t m(t) = \partial_t \int_{-\infty}^{\infty} u(x,t)\,dx = \int D(x)\, \partial_x^2 u(x,t)\,.dx$$

$$= -\int \partial_x D \cdot \partial_x u \cdot dx$$

$$\boxed{\partial_t m(t) = \epsilon\, \partial_x u\big(X(t),t\big)}$$



$$X(t) = \sqrt{t+\ell^2} + O(\epsilon)$$

SUBSTITUTE SOLUTION

$$\partial_t m = -\frac{\epsilon\, m(t)}{\sqrt{2\pi}}\, \frac{e^{-1/2}}{(t+\ell^2)}$$

$$\boxed{\begin{aligned} m(t) &= m(0)\, \frac{\ell^{2\alpha}}{(t+\ell^2)^{\alpha}} \\ \alpha &= \epsilon/\sqrt{2\pi e} \end{aligned}}$$

HEURISTIC DERIVATION (2)

SOLUTION IN FORM

$$u(x,t) = \frac{m(t)}{\sqrt{2\pi(t+\ell^2)}} e^{-x^2/2(t+\ell^2)}$$

TIME VARIATION OF MASS

$$m(t) \cong m(0) \frac{\ell^{2\alpha}}{(t+\ell^2)^{\alpha}}, \qquad \alpha = \frac{\epsilon}{\sqrt{2\pi e}}$$

SOLUTION

$$\boxed{u(x,t) = \frac{m(0)\,\ell^{2\alpha} \cdot e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi}\,(t+\ell^2)^{\frac{1}{2}+\alpha}}}$$

- MORE CAREFUL RENORMALISATION GROUP ANALYSIS SHOWS THAT

$$\alpha = \frac{\epsilon}{\sqrt{2\pi e}} - 0.101\cdots\epsilon^2 + O(\epsilon^3)$$

AND FORM OF $u(x,t)$ CORRECT TO $O(\epsilon)$.

- EXPANSION FOR $\alpha(\epsilon)$ IS ANALYTIC (ARONSON + VAZQUEZ)
- LIMIT $\ell \to 0$ SINGULAR
- NO NOISE IN BARENBLATT EQN OR PARTITION FUNCTION

INTERPRETATION

$\epsilon = 0$ MEASUREMENT AT LONG TIMES OF $m(t)$ IMPLIES KNOWLEDGE OF INITIAL VALUE $m(0)$.
$l \to 0$ LIMIT O.K. SYSTEM "FORGETS" INITIAL CONDITION AFTER SUFFICIENTLY LONG TIME. $u(x,t) \xrightarrow[\frac{t}{l^2} \to \infty]{} \frac{m e^{-x^2/2t}}{(2\pi t)^{1/2}}$

$\epsilon \neq 0$ AT LATE TIMES CANNOT INFER $m(0)$ FROM $m(t)$ ALONE. INDEED, ONE CANNOT EVEN TELL HOW MUCH TIME HAS ELAPSED!
$l \to 0$ LIMIT SINGULAR. SYSTEM "REMEMBERS" EXISTENCE OF INITIAL CONDITION WITH NON-ZERO WIDTH. BUT ANOMALOUS DIMENSION IS INDEPENDENT OF l. $u(x,t) \xrightarrow[\frac{t}{l^2} \to \infty]{} \frac{m(0)\, l^{2\alpha}\, e^{-x^2/2t}}{(2\pi)^{1/2}\, t^{1/2+\alpha}}$

ANOMALOUS DIMENSIONS AT CRITICAL POINTS

TWO POINT CORRELATION FUNCTION

$$\boxed{G(\underline{x}-\underline{y}) = \langle \phi(\underline{x})\, \phi(\underline{y}) \rangle}$$

ORDER PARAMETER

AT $T = T_c$

$$\boxed{\hat{G}(k, T_c) \sim k^{-2+\eta}} \quad \circledast$$

ANOMALOUS DIMENSION

BUT DIMENSIONAL ANALYSIS GIVES

$$\boxed{[\phi] = L^{1-d/2} \Rightarrow [\hat{G}(k,T_c)] = L^2}$$

DIMENSION OF ....

LENGTH

DIMENSIONALITY OF SPACE

DOES $\circledast$ VIOLATE DIMENSIONAL ANALYSIS?

NO! MUST INCLUDE LATTICE SPACING ℓ

$$\boxed{\hat{G}(k,T_c) \sim \ell^{\eta} k^{-2+\eta}}$$

EVEN WHEN CORRELATION LENGTH $\to \infty$, SYSTEM "REMEMBERS" EXISTENCE OF LATTICE.

PERTURBATIVE RENORMALISATION

1. WRITE BARENBLATT EQUATION AS

$$\left[\partial_t - \tfrac{1}{2}\partial_x^2\right] u(x,t) = \frac{\epsilon}{2}\,\Theta\big(X(t,\epsilon) - |x|\big)\,\partial_x^2 u$$

WHERE $\quad \partial_t u(X(t,\epsilon), t) = 0$

2. SOLUTION IS

$$u(x,t) = \int_{-\infty}^{\infty} dy\, G(x-y,t)\, u(y,0)$$

$$+ \frac{\epsilon}{2}\int_0^t ds \int_{-\infty}^{\infty} dy\, G(x-y,t-s)\,\Theta\big(-\partial_s u(y,s)\big)\,\partial_y^2 u(y,s)$$

$G(x,y) \equiv \frac{1}{\sqrt{2\pi y}}\, e^{-x^2/2y}$

3. EVALUATION OF INTEGRALS AND ISOLATION OF DIVERGENCES GIVES

$$\boxed{u(x,t) = \frac{m_0}{\sqrt{2\pi t}}\, e^{-x^2/2t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{\ell^2} + O(\epsilon^2)\right] + O(\ell,\epsilon)}$$

MASS ASSOCIATED WITH INITIAL CONDITION OF WIDTH ℓ

REGULAR AS $\ell \to 0$

4. PERTURBATIVE RENORMALISATION

IN LIMIT $\ell \to 0$ $\quad m_0$ MAY GO TO ZERO OR INFINITY.
BUT DISTRIBUTION MASS AT TIME t STILL EXISTS AND IS OBSERVABLE.

$$m = Z^{-1}\left(\frac{\ell}{\mu}, \epsilon\right) m_0$$

N.B. $m = m(\mu, \epsilon)$

Z IS DIMENSIONLESS $\Longrightarrow$ CANNOT DEPEND ON ℓ ALONE:
NEED ANOTHER LENGTH SCALE μ.

PERTURBATIVE RENORMALISATION (2)

5. POWER SERIES EXPANSION OF Z

$$Z = 1 + \sum_{n=1}^{\infty} a_n(\ell/\mu)\,\epsilon^n$$

CHOSE a_n ORDER BY ORDER IN ϵ SO THAT $u(x,t)$ IS FINITE

$$a_1(\ell/\mu) = \frac{1}{\sqrt{2\pi e}} \log\left(\frac{C_1\mu^2}{\ell^2}\right), \quad C_1 \text{ arbitrary}$$

$$u(x,t) = \frac{m}{\sqrt{2\pi t}} e^{-x^2/2t}\left[1 + \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{C_1\mu^2}{\ell^2} + O(\epsilon^2)\right] \times$$

$$\times\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{\ell^2} + O(\epsilon^2)\right] + O(\ell,\epsilon)$$

6. CANCELLATION OF DIVERGENCE AS $\ell \to 0$

MULTIPLYING OUT $[\cdots]\times[\cdots]$

$$= 1 - \frac{\epsilon}{\sqrt{2\pi e}}\left[\log\frac{t}{\cancel{\ell^2}} + \log\frac{\cancel{\ell^2}}{C_1\mu^2} + O(\epsilon)\right]$$

7. RENORMALIZATION GROUP EQUATION

THE SCALE μ IS ARBITRARY SO $u(x,t)$ INDEPENDENT OF μ.

$$\frac{du}{d\mu} = 0 \quad\Rightarrow\quad \frac{dm}{m} = -\frac{2\epsilon}{\sqrt{2\pi e}}\frac{d\mu}{d\mu} \quad\Rightarrow\quad m \sim m_0\,\mu^{-2\epsilon/\sqrt{2\pi e}}$$

i.e. $m(\mu) = m(\sigma)\left(\frac{\sigma}{\mu}\right)^{2\epsilon/\sqrt{2\pi e}}$

RELATES m AT TWO DIFFERENT SCALES μ AND σ

PERTURBATIVE RENORMALISATION (3)

8. ELIMINATE LOG TERM BY SUITABLE CHOICE OF μ

$$u(x,t) = m(\mu)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\left(1-\frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t}{C_1\mu^2}+O(\epsilon^2)\right)+O(\cancel{\ell},\epsilon)$$

CHOSE $\mu^2 = t/C_1 \Rightarrow \log t/C_1\mu^2 = 0$

$$u(x,t) = m_0\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\left(\frac{C_1}{t}\right)^{\epsilon/\sqrt{2\pi e}}\left(1+O(\epsilon^2)\right)+O(\epsilon)$$

$$\boxed{u(x,t) \sim t^{-(\alpha+1/2)} \qquad \alpha = \frac{\epsilon}{\sqrt{2\pi e}}+O(\epsilon^2)}$$

9. APPLY INITIAL CONDITIONS

KEEPING FACTORS OF ℓ

$$u(x,t) = m(\mu)\frac{e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi(t+\ell^2)}}\left(1-\frac{\epsilon}{\sqrt{2\pi e}}\log\frac{t+\ell^2}{C_1\mu^2}+O(\epsilon^2)\right)+O(\epsilon)$$

AT $t=0$ $\quad \mu = \sigma \equiv \ell/\sqrt{C_1}$ AND $m(\sigma) = m_0$

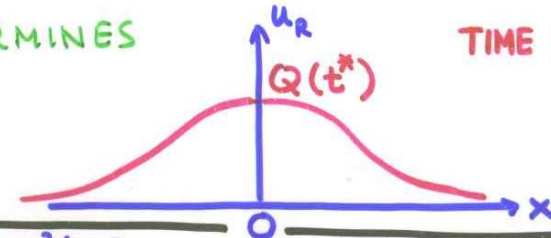
$$\boxed{u(x,t) = m_0\left(\frac{\ell^2}{t}\right)^{\epsilon/\sqrt{2\pi e}}\frac{e^{-x^2/2(t+\ell^2)}}{\sqrt{2\pi(t+\ell^2)}}}$$

ANOMALOUS DIMENSIONS (5)

(d) CHOSE ONE FROM FAMILY OF SOLUTIONS

THIS DETERMINES $C_1, \ldots$

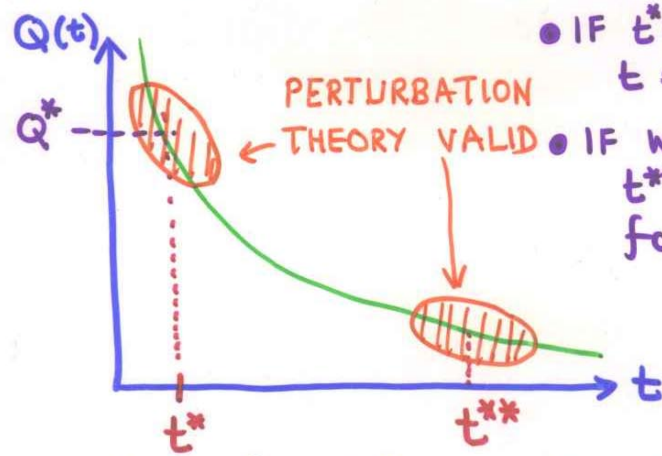
TIME t^*



$$\boxed{u_R(x,t) = Q(t^*)\sqrt{\frac{t^*}{t}}\, e^{-x^2/2t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \ln\left(\frac{t}{t^*}\right) + O(\epsilon^2)\right]}$$

THIS PERTURBATIVE SOLUTION VALID FOR $t \approx t^*$.

(e) BUT HAVE NOT YET SPECIFIED $t^*, Q(t^*)$



- IF $t^* = 5$ secs, PT poor for $t = 10^6$ secs.
- IF WE KNEW $Q(t^{**})$, WITH $t^{**} = 9 \times 10^5$ secs., PT good for $t = 10^6$ secs.

$\Rightarrow$ CHOSE $t^* = 9 \times 10^5$ secs. BUT INSIST THAT u_R STAYS ON THE PARTICULAR SOLUTION WITH $Q = Q^*$ AT $t = 5$ secs.

ANOMALOUS DIMENSIONS (6)

(f) GELLMANN–LOW TRICK:

$u_R(x,t)$ IS INDEPENDENT OF t^*.

$$\Rightarrow \boxed{\frac{\partial u_R}{\partial t^*} + \frac{\partial u_R}{\partial Q}\frac{dQ}{dt^*} = 0}$$

$$\therefore \beta(Q) \equiv t^* \frac{dQ}{dt^*} = \frac{-t^* \frac{\partial u_R}{\partial t^*}}{\frac{\partial u_R}{\partial Q}}$$

$$\boxed{\beta(Q) = -Q\left[\tfrac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)\right]}$$

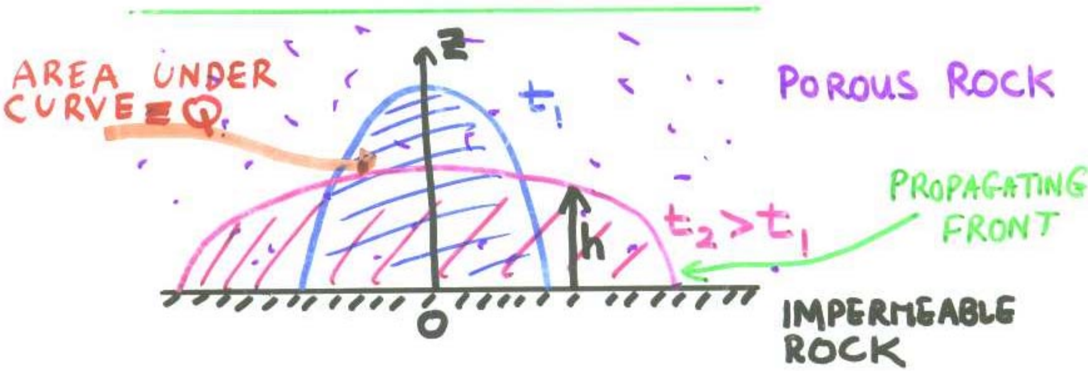
(g) INTEGRATE β - FUNCTION

$$Q(t^*) = (At^*)^{-\left[\frac{1}{2} + \epsilon/\sqrt{2\pi e} + O(\epsilon^2)\right]}$$

SET $t^* = t$:

$$\boxed{\begin{aligned} U_R(x,t) &= \frac{A}{t^{\frac{1}{2}+\alpha}} e^{-x^2/2t}\left(1 + O(\epsilon^2)\right) \\ \alpha &= \epsilon/\sqrt{2\pi e} + O(\epsilon^2) \end{aligned}}$$

GROUNDWATER SPREADING



GROUNDWATER MOUND SPREADS DUE TO GRAVITY

PHYSICS — D'ARCY LAW FLOW

— INITIALLY FILLED PORE DOES NOT DRAIN COMPLETELY DUE TO WETTING

$$\boxed{\begin{aligned} \partial_t h &= \kappa \frac{1}{r} \partial_r \left(r \partial_r h^2 \right) \\ \kappa &= \begin{cases} 1 & \partial_t h > 0 \\ 1+\epsilon & \partial_t h < 0 \end{cases} \end{aligned}}$$

h cts

$\partial_r h$ cts for $h \neq 0$

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GROUNDWATER SPREADING (2)

DIMENSIONAL ANALYSIS $\Rightarrow$

$$h = h(Q, x, t, r, \ell, \epsilon)$$

$[h] = H$; $[Q] = HL^2$; $[t] = T$; $[r] = [\ell] = L$;

$[\kappa] = L^2 T^{-1} H^{-1}$; $[\epsilon] = 1$

$$\therefore \boxed{h = \frac{Q^{1/2}}{(\kappa t)^{1/2}} \Phi\left(\frac{r}{[Q\kappa t]^{1/4}}, \frac{\ell}{[Q\kappa t]^{1/4}}, \epsilon\right)}$$

THE GROUNDWATER EQUATION DOES NOT CONSERVE

$$Q = \int 2\pi r h(r,t)\, dr$$

$\Rightarrow$ KNOWING Q AT A LATE TIME DOES NOT IMPLY KNOWLEDGE OF Q AT EARLY TIME.

i.e. CAN REGARD $Q = Q(t)$ or $Q(\ell)$. FOR EACH INITIAL CONDITION WITH WIDTH ℓ, THERE IS A VALUE OF Q : BUT $\lim_{\ell \to 0} Q(\ell)$ NOT WELL DEFINED. NEVERTHELESS, THERE IS A MEASURABLE QUANTITY Q AT $\lim_{t \to \infty} = \lim_{\ell \to 0}$.

PHENOMENOLOGICAL PARAMETER

$$\boxed{Q = \underbrace{Z\left(\frac{\ell}{\mu}\right)}_{[Z] = 1} Q(\ell)}$$

$[Z] = 1$

REQUIRED BY D.A.
ARBITRARY LENGTH

GROUNDWATER SPREADING (3)

∴ CAN WRITE PHENOMENOLOGICAL EQUATION

$$\boxed{h = \frac{Z^{1/2} Q^{1/2}}{(\kappa t)^{1/2}} \Phi\left(\underbrace{\frac{r}{Z^{1/4}(Q\kappa t)^{1/4}}}_{\xi}, \underbrace{\frac{\mu}{Z^{1/4}(Q\kappa t)^{1/4}}}_{\eta}, \epsilon \right)}$$

THE ACTUAL SOLUTION CANNOT DEPEND ON THE ARBITRARY PARAMETER μ.

$$\mu \frac{dh}{d\mu} = 0$$

$$\Rightarrow -\frac{1}{2}\alpha \Phi + \frac{1}{4}\alpha \xi \frac{\partial \Phi}{\partial \xi} + \left(1 + \frac{\alpha}{4}\right) \eta \frac{\partial \Phi}{\partial \eta} = 0$$

where $\alpha \equiv - \dfrac{\partial \log Z}{\partial \log \mu}$

Solve (i.e. by method of characteristics, ...)

$$\Phi = \eta^{\beta} f\left(\xi \eta^{-\beta/2}, \epsilon \right)$$

SCALING LAW!

$$\beta = \frac{\alpha/2}{1 + \alpha/4}$$

$$\Rightarrow \boxed{h(r,t) = \frac{1}{t^{1/2 + a}} f\left(\frac{r}{t^{1/4 + b}}, \epsilon \right)}$$

$a = -2b = \beta/4$ (c.f. Barenblatt, D.A. p.123)

STRONG THERMAL WAVE

GENERALISATION OF GROUNDWATER EQUATION:

$$\boxed{\begin{aligned} \partial_t u &= \kappa \frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r u^{1+n} \right) \\ \kappa &= \begin{cases} 1 & \partial_t u > 0 \\ 1+\epsilon & \partial_t u < 0 \end{cases} \end{aligned}}$$

CONVENIENT TO TRANSFORM:

$$v \equiv \frac{n+1}{n} u^n$$

$$\Rightarrow \quad \boxed{\partial_t v = \kappa \left[(\underline{\nabla} v)^2 + n v \Delta v \right]}$$

Naïve perturbation theory:

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots$$

$$\boxed{v_0(r,t) = \frac{\theta}{2} \frac{Q^n \xi_0^{1/\theta}}{s^{nd/2}} \left(1 - \frac{r^2}{s} \right)} \quad \left(r^2/s < 1 \right)$$

$$s \equiv \left(Q^n t \xi_0^{1/\theta} + \ell^{1/\theta} \right)^{2\theta}$$

$$\theta \equiv \frac{1}{2+nd} \qquad \xi_0 \equiv \left\{ S_d \left[\frac{n\theta}{2(n+1)} \right]^{1/n} \int_0^1 s^{d-1} (1-s^2)^{1/n} ds \right\}^{-n\theta}$$

STRONG THERMAL WAVE (2)

EQUATION FOR v_1:

$$\boxed{\left[s\partial_s - nL\right]v_1 = \frac{\theta}{2}\frac{Q_0^n s_0^{1/\theta}}{s^{nd/2}}\left[-\frac{nd}{2} + y\left(1+\frac{nd}{2}\right)\right] \times \Theta\left(\sqrt{nd\theta} - \sqrt{y}\right)}$$

Heaviside step function

$$y \equiv r^2/s$$

$$L \equiv y(1-y)\frac{d^2}{dy^2} + \left[\frac{d}{2} - \left(\frac{d}{2} + \frac{1}{n}\right)y\right]\frac{d}{dy} - \frac{d}{2}$$

Eigenfunctions of L are Jacobi polynomials.

SOLVE BY GREEN FUNCTIONS:

$\exists$ log divergence as $t/\ell^{1/\theta} \to \infty$

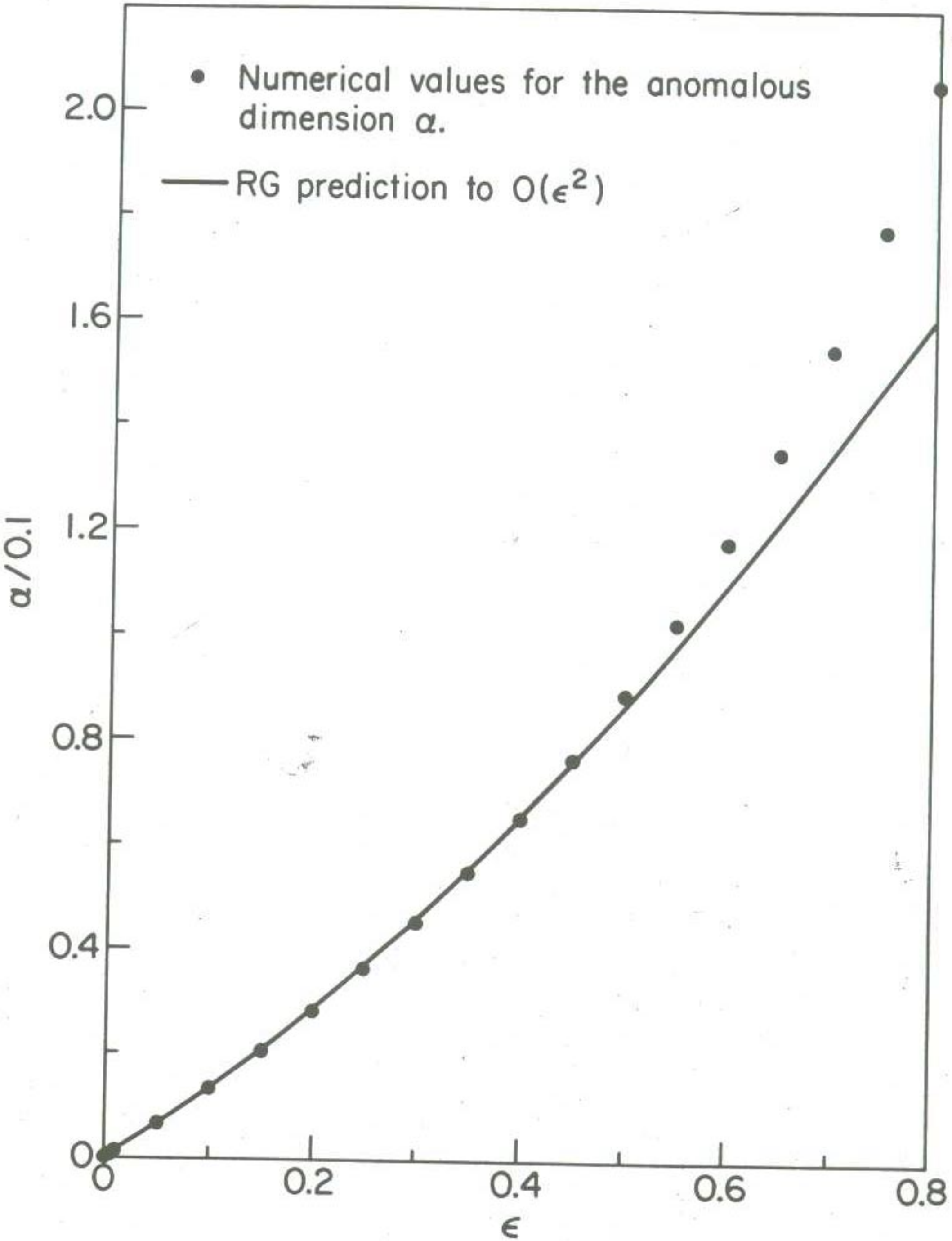
RG calculation $\Longrightarrow$

$$\boxed{u(r,t) \sim \left[\frac{A}{(Q_0^n t)^{nd\theta+\alpha}} - \frac{n\theta}{2(n+1)}\frac{r^2}{t}\right]^{1/n}}$$

$$\boxed{\alpha = \epsilon\lambda + O(\epsilon^2)}$$

$$\lambda = \frac{4}{d(d+2)}\frac{\Gamma\left(\frac{d}{2}+\frac{1}{n}\right)}{\Gamma(1/n)\Gamma(d/2)}(nd\theta)^{1+\frac{d}{2}} F\left(1-\frac{1}{n}, \frac{d}{2}; \frac{d}{2}+2; nd\theta\right)$$

HYPERGEOMETRIC FN.



Numerical values for the anomalous dimension α.
RG prediction to $O(\epsilon^2)$
$\alpha/0.1$
0
0.4
0.8
1.2
1.6
2.0
0
0.2
0.4
0.6
0.8
ϵ

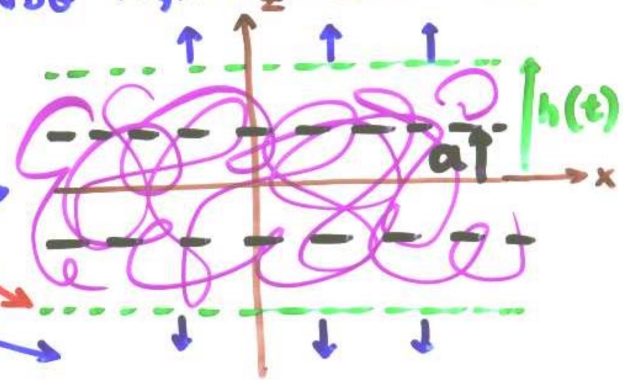
PROPAGATION OF TURBULENCE

1. THEORY

PROPAGATION OF TURBULENCE

REF: G.I. Barenblatt in Nonlinear Dynamics and Turbulence, G. Barenblatt, Ioos, Joseph (eds.) (Pitman 1983).

L. Y. Chen, NDG Phys. Rev. A 45, 5572 (92)



1. FORMULATION

PROBLEM: DESCRIBE PROPAGATION OF THE TURBULENT BURST INTO QUIESCENT FLUID

PHYSICS: SHEARLESS FLOW OF INCOMPRESSIBLE FLUID. ENERGY DISSIPATED AS HEAT. BALANCE OF TURBULENT ENERGY.

$$\boxed{\partial_t q = \partial_z (K \partial_z q) - E_t}$$

mean kinetic energy per unit mass (q)

turbulent eddy diffusion coefficient (K)

mean rate of turbulent energy dissipation per unit mass (E_t)

$$\boxed{|z| \leq h(t)}$$

PROPAGATION OF TURBULENCE (2)

KOLMOGOROV SIMILARITY HYPOTHESIS

K, E_t functions of mean eddy size ℓ, local eddy mean energy.

$$\Rightarrow \quad \boxed{K = \ell\sqrt{q}} \; ; \quad \boxed{E_t = \frac{\epsilon q^{3/2}}{\ell}} \quad (\epsilon > 0)$$

ALSO ASSUME THAT ℓ IS A FIXED FRACTION OF THICKNESS OF TURBULENT LAYER

$$\boxed{\ell = \alpha h(t) \qquad \alpha < 1}$$

INITIAL CONDITIONS:

FIXED AMOUNT OF ENERGY IN BURST OF THICKNESS a.

$$q(z,0) = \frac{Q}{a} u\left(\frac{z}{a}\right) \; ; \quad Q_a = \int_{-a}^{a} q(z,0)\,dz$$

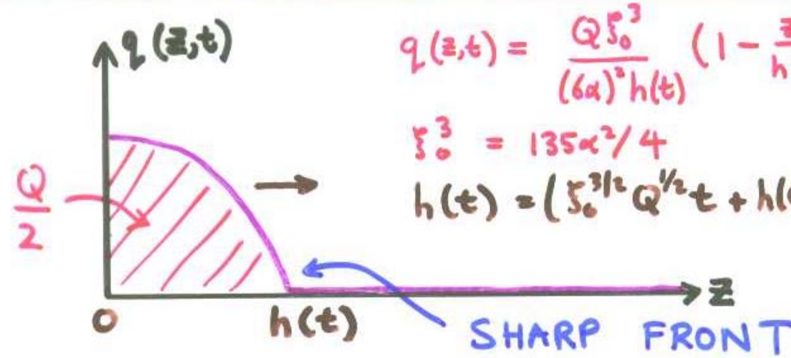
$$\int_{-1}^{1} u(\tau)\,d\tau = 1$$

$$\partial_t q = \alpha \partial_z\left(h(t) q^{\frac{1}{2}} \partial_z q\right) - \frac{\epsilon q^{3/2}}{\alpha h(t)} \qquad |z| \le h(t)$$

$$q = 0 \qquad |z| > h(t)$$

PROPAGATION OF TURBULENCE (3)

2. SOLUTION IN ABSENCE OF DISSIPATION



$$q(z,t) = \frac{Q\xi_0^3}{(6\alpha)^2 h(t)}\left(1 - \frac{z^2}{h(t)^2}\right)_+^2$$

$$\xi_0^3 = 135\alpha^2/4$$

$$h(t) = \left(\xi_0^{3/2} Q^{1/2} t + h(0)^{3/2}\right)^{2/3} \sim t^{2/3}$$

3. DISSIPATION : HEURISTIC CALCULATION

Q NO LONGER CONSERVED

$$\dot{Q} \equiv \partial_t \int_{-h(t)}^{h(t)} q(z,t)\,dz = -\frac{\epsilon}{\alpha h(t)}\int_{-h}^{h} q^{3/2}\,dz < 0$$

FOR SLOW REMOVAL OF ENERGY c.f. PROPAGATION RATE

$$\dot{Q} = -\frac{\epsilon}{\alpha}\frac{Q^{3/2}}{(6\alpha)^3}\frac{\xi_0^{9/2}}{h^{3/2}}\int_{-1}^{1}(1-x^2)^3\,dx$$

FOR LONG TIMES $h(t) \gg h(0)$ $h \sim \xi_0 Q^{1/3} t^{2/3}$

$$Q(t) = Q(t_0)\left(\frac{t}{t_0}\right)^{-\epsilon/7\alpha^2}$$

$$\boxed{h(t) \sim t^{2/3 - \underbrace{\epsilon/21\alpha^2 + O(\epsilon^2)}_{\text{ANOMALOUS DIMENSION}}}}$$

PROPAGATION OF TURBULENCE (3)

THIS IS EXPLICITLY A MOVING BOUNDARY PROBLEM: IN SOLVING FOR $q(z,t)$ WE MUST ALSO DETERMINE THE THICKNESS OF THE TURBULENT SLAB $h(t)$.

2. DIMENSIONAL ANALYSIS

$[q] = L^2 T^{-2}$; $[Q_a] = L^3 T^{-2}$

$[z] = [a] = [h] = L$

$$\boxed{\begin{aligned} q(z,t) &= \frac{Q_a^{2/3}}{t^{2/3}} f\left(\frac{z}{Q_a^{1/3} t^{2/3}}, \frac{a}{Q_a^{1/3} t^{2/3}} ; \alpha, \epsilon\right) \\ h(t) &= Q_a^{1/3} t^{2/3} F\left(\frac{a}{Q_a^{1/3} t^{2/3}} ; \alpha, \epsilon\right) \end{aligned}}$$

f, F ARE DIMENSIONLESS SCALING FUNCTIONS

WHEN $\epsilon = 0$ (NO DISSIPATION) THERE IS A SIMILARITY SOLUTION OBTAINED BY SETTING $a = 0$.

$$\boxed{\begin{aligned} h(t) &= \xi_0(\alpha)\, Q^{1/3} t^{2/3} \\ \xi_0(\alpha) &= \left[135\,\alpha^2/4\right]^{1/3} \end{aligned}}$$

PROPAGATION OF TURBULENCE (4)

3. DISSIPATION : $\epsilon \neq 0$

NOW WE ALLOW FOR THE POSSIBILITY THAT THE LIMIT $a \to 0$ MAY NOT BE WELL-DEFINED. NEVERTHELESS, PHYSICALLY, WE ARE INTERESTED IN THE ASYMPTOTIC LONG-TIME BEHAVIOUR, WITH $h(t) \gg a$.

AT LONG TIMES, Q_a, THE INITIAL ENERGY PER UNIT MASS, IS NOT NECESSARILY MEASURABLE, BECAUSE THERE IS DISSIPATION. HENCE, WE INTRODUCE A PHENOMENOLOGICAL Q:

$$\boxed{Q = Z^{-1}\left(\frac{a}{\mu}\right) Q_a}$$

RENORMALISATION CONSTANT

ARBITRARY LENGTH

$$\boxed{q(z,t) = \frac{(ZQ)^{2/3}}{t^{2/3}}\, \bar{f}\left(\underbrace{\frac{z}{(ZQ)^{1/3}t^{2/3}}}_{\to\,\xi}, \underbrace{\frac{\mu}{(ZQ)^{1/3}t^{2/3}}}_{\to\,\eta}, \frac{a}{\mu}; \alpha, \epsilon\right)}$$

RENORMALISABILITY: ALL SINGULAR BEHAVIOUR ACCOUNTED FOR BY Z.

$\bar{f}$ smooth as this parameter $\to 0$

PROPAGATION OF TURBULENCE (5)

4. RENORMALISATION GROUP

q CANNOT DEPEND ON μ SO

$$\boxed{\mu \frac{dq}{d\mu} = 0}$$

TAKING THE LIMIT $a \to 0$ AND ASSUMING THE EXISTENCE OF THE LIMIT

$$\boxed{\gamma \equiv \lim_{a\to 0} - \frac{d \log Z}{d \log \mu}}$$

$$\boxed{\left[\frac{\gamma}{3}\xi\frac{\partial}{\partial \xi} + \left(1+\frac{\gamma}{3}\right)\eta\frac{\partial}{\partial \eta} - \frac{2\gamma}{3}\right]\bar{f} = 0}$$

WE CAN ALSO WRITE DOWN A CORRESPONDING EQUATION FOR $h(t)$. SOLVING BY THE METHOD OF CHARACTERISTICS

$$\boxed{q(z,t) = t^{-(2/3 + A)} F\left(\frac{z}{t^{2/3+B}}\right)}$$

$$A + 2B = 0$$

$$A = \frac{4\gamma}{9+3\gamma} \; ; \quad B = \frac{-2\gamma}{9+3\gamma}$$

$$\boxed{h(t) \sim t^{2/3+B}}$$

PROPAGATION OF TURBULENCE (6)

5. PERTURBATION THEORY

PERTURBATION THEORY YIELDS DIVERGENCES FROM WHICH THE ANOMALOUS DIMENSIONS A AND B MAY BE CALCULATED, WITH SIMILAR MATHEMATICS TO GROUNDWATER PROBLEM.

$$\boxed{B = \frac{-\epsilon}{2 l q^2} + O(\epsilon^2)}$$

6. REMARKS

POSSIBLE RELEVANT VARIABLES AT THIS FIXED POINT:

- STRATIFICATION OF FLUID DUE TO TEMPERATURE (e.g.)
- BOUNDARY NOT SHARPLY DEFINED
- INITIAL FORMULATION SOMEWHAT "MEAN FIELD"-LIKE.

End of lecture 1