Collective Effects
FIGURE E1-1. Top: IBM stock from 1959 to 1996, in units of $10, plotted on logarithmic scale. Bottom: the corresponding relative daily price changes, in units of 1%.

Mandelbrot
WIND SPEED FLUCTUATIONS

Speed now - speed a moment ago

Gagne (1980)
Metals

Scale of one millionth of a meter

Air bubble

Scale of one meter

Image: David and Boatner (1997)
algebra. For convenience we use $\delta^2 = \epsilon$, and write Eq. (1) in 1-D as

$$
\left[ \partial_t + \delta^2 \left( 1 + \delta^2 \right)^2 \right] \psi = \delta^2 \partial_x^2 \psi (\psi - \psi).
$$

(30)

The basic premise of the multiple scales analysis is that while the pattern itself varies on the scale of its wavelength and time scale, it is then appropriate to introduce slowly varying arguments

$$
X - \delta x, \quad T = \delta t
$$

for the envelope function $A(X,T)$. This scaling was previously applied by Guhrat et al. [14] to the Swift-Hohenberg equation with success, and so the PCU Hohenberg equation we anticipate that the same scaling

Derivatives scale as follows

$$
\partial_t \rightarrow \partial_t + \delta \partial_x
$$

$$
\partial_x^2 \rightarrow \partial_x^2 + 2\delta^2 \partial_x + \delta \partial_x^2
$$

$$
\partial_x^4 \rightarrow \delta \partial_x^4
$$

(32)

whereas the operator

$$
\delta^2 \left( 1 + \delta^2 \right)^2 \rightarrow \sum_{j=0}^{\infty} \delta^j \partial_x^j
$$

(33)

such that

$$
\mathcal{L}_0 = \delta^2 \left( 1 + \delta^2 \right)^2
$$

$$
\mathcal{L}_1 = -4\delta^2 \partial_x^2 \left[ 1 + \delta^2 \partial_x^2 \right] + \delta \partial_x \left[ 1 + \delta^2 \partial_x^2 \right]^2
$$

$$
\mathcal{L}_2 = 12\delta^2 \partial_x^2 + 8\delta^4 \partial_x^4 \left[ 1 + \delta^2 \partial_x^2 \right] + \delta \partial_x \left[ 1 + \delta^2 \partial_x^2 \right]^2
$$

$$
\mathcal{L}_3 = \delta \partial_x \left[ 1 + \delta^2 \partial_x^2 \right]^2
$$

$$
\mathcal{L}_4 = \delta^2 \partial_x^2
$$

We now expand $\psi$ in a perturbation series in $\delta$ to get

$$
\psi = \psi_0 + \delta \psi_1 + \delta^2 \psi_2 + \delta^3 \psi_3 + \ldots.
$$

(35)

Using Eq. (32) and the above series, the $\delta$ expansion of the nonlinear term in Eq. (30) can be written as

$$
\delta^2 \left( \psi^3 - \psi \right) = \delta^2 \left( \psi_0^3 - \psi_0 \right) + \delta \left( \psi_0^3 \psi_1 + 3\psi_0 \psi_1^2 + \psi_1 \psi_0^2 \right) + \delta^2 \left( \psi_0^3 \psi_2 + 3\psi_0 \psi_1 \psi_2 + \psi_1 \psi_0 \psi_2 \right)
$$

$$
+ \delta^3 \left( \psi_0^3 \psi_3 + 3\psi_0 \psi_1 \psi_3 + \psi_1 \psi_0 \psi_3 \right) + \ldots
$$

(34)

Substituting Eqs. (30) in Eq. (31), and using the scaled operators in Eqs. (25,24), we can write equations satisfied by the $\psi_n$ at each $\mathcal{O}(n)$. At $\mathcal{O}(1)$, we obtain

$$
\mathcal{L}_0 \psi_1 = 0 \quad \Rightarrow \quad \psi_1 + \psi_1 = A_1 \psi_1 (X,T) + \ldots.
$$

(36)

where $A_1$ is the complex amplitude of mode 1 at $\mathcal{O}(1)$. At $\mathcal{O}(\delta)$ we get

$$
\mathcal{L}_1 \psi_1 + \delta \mathcal{L}_0 \psi_1 = 0 \quad \Rightarrow \quad \psi_1 + \delta \psi_1 = A_1 \psi_1 (X,T) + \ldots.
$$

(37)

where (and hence) we neglect the constant term in view of its inclusion in Eq. (36). As the next order we have

$$
\mathcal{L}_0 \psi_2 = \delta \mathcal{L}_1 \psi_1 + \delta^2 \mathcal{L}_0 \psi_1 - \mathcal{L}_2 \psi_0
$$

(38)

where $A_3 = \frac{\partial A_1}{\partial x}$ and $A_4 = \frac{\partial A_1}{\partial t}$. At subsequent orders, the following equations are obtained for $\psi_n$.

$$
\mathcal{L}_0 \psi_n = \delta \mathcal{L}_1 \psi_{n-1} - \mathcal{L}_2 \psi_{n-2} - \mathcal{L}_3 \psi_{n-3} = \ldots
$$

(39)

$$
\mathcal{L}_0 \psi_n = \delta \mathcal{L}_1 \psi_{n-1} - \mathcal{L}_2 \psi_{n-2} = \ldots
$$

(40)

$$
\mathcal{L}_0 \psi_n = \delta \mathcal{L}_1 \psi_{n-1} = \ldots
$$

(41)

$$
\mathcal{L}_0 \psi_n = \delta \mathcal{L}_1 \psi_{n-1} = \ldots
$$

(42)

We continue this process up to the $n^{th}$ order to obtain the following equations for $\psi_n$ in the equation.

$$
\left. \mathcal{L}_0 \psi_n = \delta \mathcal{L}_1 \psi_{n-1} - \mathcal{L}_2 \psi_{n-2} = \ldots \right|_{\mathcal{O}(n)}
$$

(43)

Using Eqns. (40), (43), and (44), and scaling back to original variables, i.e. $X = \delta x$ and $T = \delta t$, the amplitude equation to $\mathcal{O}(\delta^3)$ can be written as

$$
\mathcal{A}(X,T) = \mathcal{A}(X,T) + \delta \mathcal{A}_1 \psi_1 (X,T) + \ldots
$$

(44)

Using Eqns. (40), (43), and (44), and scaling back to original variables, i.e. $X = \delta x$ and $T = \delta t$, the amplitude equation to $\mathcal{O}(\delta^3)$ can be written as

$$
\mathcal{A}(X,T) = \mathcal{A}(X,T) + \delta \mathcal{A}_1 \psi_1 (X,T) + \ldots
$$

(45)

where $\mathcal{A}$ is the complex amplitude of mode $n$ at $\mathcal{O}(\epsilon^3)$. At $\mathcal{O}(\delta^3)$, we get

$$
\mathcal{L}_0 \mathcal{A} = \delta \mathcal{L}_1 \mathcal{A} + \delta^2 \mathcal{L}_2 \mathcal{A} + \ldots
$$

(46)

As such, we obtain

$$
\mathcal{A}(x,t) = \left( 1 - \delta^2 \right) \mathcal{A}_0 + \delta \mathcal{A}_1 \psi_1 (X,T) + \ldots
$$

(47)

where $\mathcal{A}_0$ is the complex amplitude of mode $n$ at $\mathcal{O}(1)$. At $\mathcal{O}(\delta)$ we get

$$
\mathcal{L}_1 \mathcal{A} + \delta \mathcal{L}_0 \mathcal{A} = 0 \quad \Rightarrow \quad \mathcal{A}_1 = \mathcal{A}_2 \psi_1 (X,T) + \ldots
$$

(48)

where $\mathcal{A}_1 = \frac{\partial \mathcal{A}_0}{\partial x}$ and $\mathcal{A}_2 = \frac{\partial \mathcal{A}_0}{\partial t}$. At subsequent orders, the following equations are obtained for $\mathcal{A}_n$.

$$
\mathcal{L}_0 \mathcal{A}_n = \delta \mathcal{L}_1 \mathcal{A}_{n-1} - \mathcal{L}_2 \mathcal{A}_{n-2} - \mathcal{L}_3 \mathcal{A}_{n-3} = \ldots
$$

(49)

$$
\mathcal{L}_0 \mathcal{A}_n = \delta \mathcal{L}_1 \mathcal{A}_{n-1} = \ldots
$$

(50)

$$
\mathcal{L}_0 \mathcal{A}_n = \delta \mathcal{L}_1 \mathcal{A}_{n-1} = \ldots
$$

(51)

We continue this process up to the $n^{th}$ order to obtain the following equations for $\mathcal{A}_n$ in the equation.
SUPER
COMPUTER
Sampling water and microbes at Yellowstone National Park

NG, Bruce Fouke

Surveying Minerva Terrace

NG, Hector Garcia Martin, John Veysey