(1

References:

N.G. Lectures on Phase Transitions and the Renormalization Group, (1992), Ch.10
G.I. Barenblatt, Scaling, Self-Similarity and Intermediate Asymptotics, Cambridge (1995).
L.Y. Chen, NG, Y. Oono, Phys Rev E 54, 376 (1996)
Q. Hou, NG, A. McKane, Phys Rev E (2001).

1. 1985 last semina

Outline.

I. Similarity Solutions and Travelling Waves.
- diffusion eqn
- dimensional analysis < type I, type II
- Barenblatt eqn.
- Heuristics of Barenblatt eqn.
- Sketch of RG
- Meaning of solution + comparison with critical phenomena
- Geometric interpretation + Wilson RG.

II. Global asymptotics
- Boundary layers
- WKB/matching
- Reductive perturbation theory.

III. Applications
- Pattern formation (e.g.) convection, particle beams,
- focusing solutions
- GR, BH.
- Phase field models
- Stochastic differential equations/turbulence
- Generalized CLT / EVT.

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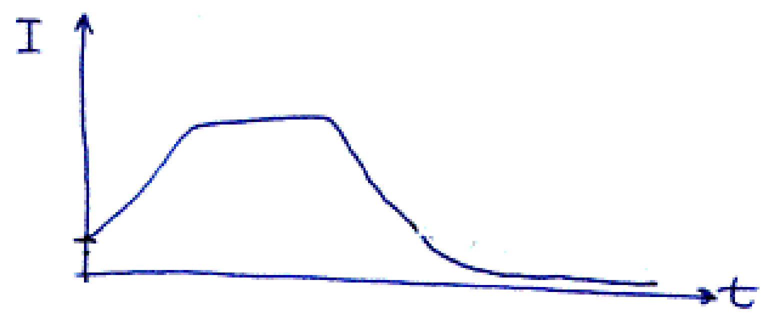

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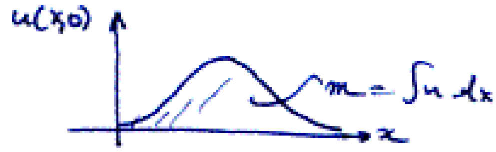
1) Diffusion.



$t=0$

$$\partial_t u = \frac{1}{2}\kappa \partial_x^2 u$$

$$u(x,0) = \frac{m}{\sqrt{2\pi l^2}} e^{-x^2/2l^2}$$

$t>0$

$$u(x,t) = \frac{m e^{-x^2/(2(l^2+\kappa t))}}{\sqrt{2\pi(l^2+\kappa t)}}$$

$t\to\infty$

$$u(x,t) \underset{t\to\infty}{\sim} \frac{m e^{-x^2/2\kappa t}}{\sqrt{2\pi\kappa t}} .$$

Same as t fixed, $l \to 0$. Asymptotic in similarity solution, from degenerate initial condition (a delta fn).

2) Dimensional analysis.

$\Pi, \Pi_0, \Pi_1, \Pi_2 \cdots \Pi_n$ dimensionless groups.

DA $\Rightarrow$ $\Pi = f(\Pi_0, \Pi_1, \cdots \Pi_n)$.

Suppose $\Pi_0 \sim 0$. ¿ $\Pi = f(0, \Pi_1, \Pi_2, \cdots \Pi_n)$. ?

No! $\lim \Pi_0 \to 0$ may not exist.

Alternatives:

(1) $\Pi = f(0, \Pi_1, \Pi_2 \cdots \Pi_n)$ "Common sense", Type I

(2) $\Pi = \Pi_0^{\alpha} f\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, \frac{\Pi_2}{\Pi_0^{\alpha_2}} \cdots \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right)$ Type II.

(3) None of the above.

Example. Diffusion eqn.

$$\Pi = \frac{u}{m}\sqrt{\kappa t} \qquad \Pi_1 = \frac{x}{\sqrt{\kappa t}} \qquad \Pi_0 = \frac{l}{\sqrt{\kappa t}}$$

For $\Pi_0 \sim 0$ as $t\to\infty$ or $l \to 0$ we use common sense

$\Pi = f(\Pi_0, \Pi_1) \sim \tilde{f}(\Pi_1) \Rightarrow u = \frac{m}{\sqrt{\kappa t}} f(x/\sqrt{\kappa t})$.

Solve for $\tilde{f}'' + \xi \tilde{f}' + \tilde{f} = 0$ $\quad \tilde{f} \to 0 \ |x| \to \infty$. $\quad \xi \equiv x/\sqrt{\kappa t}$

$\tilde{f} = e^{-\xi^2/2}$

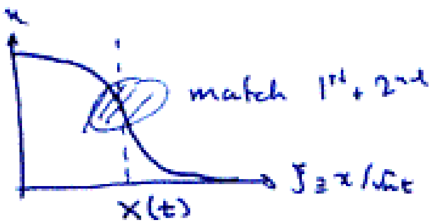
③

③ Barenblatt equation

$$\partial_t u(x,t) = D\kappa\, \partial_x^2 u \qquad D = \begin{cases} \frac{1}{2} & \partial_t u \geq 0 \\ \frac{1}{2}(1+\epsilon) & \partial_t u < 0 \end{cases}$$

DA: $\Pi = \frac{u}{m}\sqrt{\kappa t}$ $\quad \Pi_0 = \frac{\ell}{\sqrt{\kappa t}}$ $\quad \Pi_1 = \frac{x}{\sqrt{\kappa t}}$ $\quad \Pi_2 = \epsilon.$

Guess: $\quad u = \frac{m}{\sqrt{\kappa t}} \tilde{f}\left(\frac{x}{\sqrt{\kappa t}}, \epsilon\right) \qquad t \to \infty.$



match 1^{st}, 2^{nd} derivs. Can't do it. Silly eqn? No!
Kamenomostkaya (1958) proved existence and uniqueness with continuous 2nd derivs.

Actual solution.

$$u(x,t) = \frac{1}{t^{\frac{1}{2}+\alpha(\epsilon)}} \bar{f}\left(\frac{x}{\sqrt{\kappa t}}, \epsilon\right)$$

$\alpha(\epsilon)$ = anomalous dimension

④ Heuristic solution.

Let's understand physics of Barenblatt term. B-eqn is not local conservation law

$$\partial_t u = -\underline{\nabla} \cdot \underline{J}$$

$m \equiv \int_{-\infty}^{\infty} u(x,t)\,dx$ not constant of motion. i.e. $m = m(t)$.

Suppose mass removed slowly w.r.t. spreading. Then in this adiabatic limit, profile adjusts to distribution when D = constant and

$$u = \frac{m(t)\, e^{-x^2/2(\kappa t+\ell^2)}}{\sqrt{2\pi(\kappa t+\ell^2)}}.$$

How to find $m(t)$?

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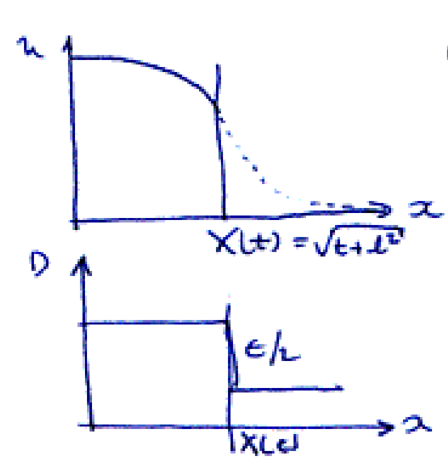

$$m(t) = \int_{-\infty}^{\infty} u \, dx$$

$$\partial_t m = \int \partial_t u = \int D(x) \partial_x^2 u$$

$$= -\int \partial_x D \partial_x u = \epsilon\kappa \partial_x u(X(t), t)$$

$$= -\frac{\epsilon \kappa m(t)}{\sqrt{2\pi}} \frac{e^{-1/2}}{(\kappa t + l^2)}$$

Solution: $m(t) = m(0) \dfrac{l^{2\alpha}}{(\kappa t + l^2)^\alpha}$ $\quad \alpha = \epsilon/\sqrt{2\pi e}$.

Long time behavior.

$$\boxed{u(x,t) \sim \frac{m(0)\, l^{2\alpha}}{(\kappa t)^{1/2 + \alpha(\epsilon)}} e^{-x^2/2\kappa t} \qquad \alpha = \epsilon/\sqrt{2\pi e}}$$

Comments.

$\epsilon = 0$ $\quad u(x,t) \underset{t\to\infty}{\sim} \dfrac{m(0)\, e^{-x^2/2\kappa t}}{(\kappa t)^{1/2}}$ $\quad$ independent of l.

$\epsilon \neq 0$ $\quad u(x,t) \underset{t\to\infty}{\sim} \dfrac{m(0)\, l^{2\alpha}\, e^{-x^2/2\kappa t}}{(\kappa t)^{\frac{1}{2}+\alpha(\epsilon)}}$ $\quad$ depends on l. Long-term memory.

c.f. critical phenomena

$$G(k) = \langle |\phi|_k^2 \rangle \sim k^{-2+\eta}$$

But $[\phi] = L^{1-d/2} \Rightarrow [G(k)] = L^2$.

What happened? Needed ultraviolet cut-off (lattice) : $G(k) \sim l^\eta k^{-2+\eta}$

Renormalisation in space $\longleftrightarrow$ renormalisation in time.

5 RG: Gell-Mann Low

Step 1 $\quad \left(\partial_t - \frac{1}{2}\kappa \partial_x^2\right) u = \frac{\epsilon}{2} \Theta\left(X(t,\epsilon) - |x|\right) \partial_x^2 u$

$$\partial_t u(X(t), t) = 0$$

Step 2

$$u(x,t) = \int G(x-y,t)\,u(y,0) + \frac{\epsilon}{2}\int_0^t ds \int dy\, G(x-y,t-s)\,\Theta(-\partial_t u(y,s))\,\partial_y^2 u($$

$$G(x,y) \equiv \frac{1}{\sqrt{2\pi y}}\, e^{-x^2/2y}$$

Step 3 Evaluate integrals and isolate divergences

$$u(x,t) = \frac{m_0}{\sqrt{2\pi \kappa t}} e^{-x^2/2\kappa t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{\kappa t}{\ell^2} + O(\epsilon^2)\right] + O(\ell,\epsilon)$$

Step 4 Perturbative renormalisation

- m_0 is initial mass.
- At time t, mass is lower because B-eqn doesn't conserve mass.
- Diffusion operator does not recognise this. The divergence of pert. theory is its way of telling us that something singular is going on.
- m_0 is in fact m at time t:

$$m = Z^{-1}(\ell/\mu, \epsilon)\, m_0$$

Step 5 Power series expansion

$$Z = 1 + \sum_{n=1}^{\infty} a_n(\ell/\mu)\,\epsilon^n$$

Chose a_n order by order in pert. theory to remove divergences.
Here we see that we should chose

$$a_1(\ell/\mu) = \frac{1}{\sqrt{2\pi e}}\log\left(\frac{C_1\mu^2}{\ell^2}\right) \qquad C_1 \text{ arbitrary} > 0.$$

$$u(x,t) = \frac{m}{\sqrt{2\pi\kappa t}} e^{-x^2/2\kappa t}\left(1 + \frac{\epsilon}{\sqrt{2\pi e}}\log\frac{C_1\mu^2}{\ell^2} + O(\epsilon^2)\right)\left(1 - \frac{\epsilon}{\sqrt{2\pi e}}\log t/\ell^2 + O(\epsilon^2)\right) + O(\ell,\epsilon)$$

Step 6 Cancel divergence.

$$[\quad] \times [\quad] = 1 - \frac{\epsilon}{\sqrt{2\pi e}}\left(\log\frac{t}{\ell^2} + \log\frac{\ell^2}{C_1\mu^2}\right)$$

Step 7 RG eqn. Let's write this in time domain. This is family of solution. Let's specify that at some time t^*, $u(0,t^*)$ has value $Q(t^*)$. Then.

$$u = Q(t^*)\sqrt{\frac{t^*}{t}}\, e^{-x^2/2t}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}}\log t/t^* + O(\epsilon^2)\right]$$

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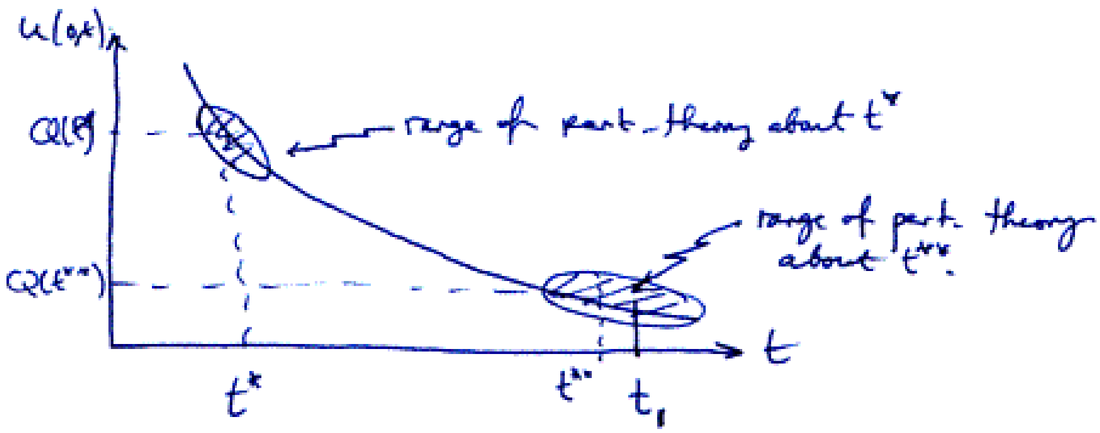


RG eqn: $t^* \frac{du}{dt^*} = 0$.

i.e. $\frac{\partial u}{\partial t^*} + \frac{dQ}{dt^*}\frac{\partial u}{\partial Q} = 0 \Rightarrow t\frac{dQ}{dt} = -Q\left[\frac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)\right]$

$\Rightarrow \quad Q(t) \sim t^{-\left(\frac{1}{2} + \alpha(\epsilon)\right)}$

Step 8. Plug into renormalized pert. expansion and set $t = t^*$.

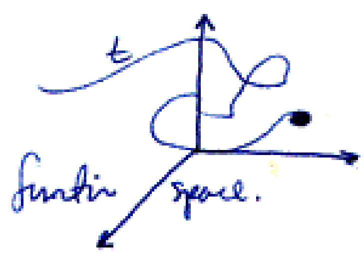
$$u \sim \frac{e^{-x^2/2\kappa t}}{t^{\frac{1}{2}+\alpha(\epsilon)}}$$

$$u = m_0 \left(\frac{\ell^2}{\kappa t}\right)^{\epsilon/\sqrt{2\pi e}} \frac{e^{-x^2/2(\kappa t)}}{\sqrt{2\pi\kappa t}}$$

6 Wilson RG

Define rescaling transformation

$$u'(x, t_0) \equiv R_{b,\phi}\left[u(x, t_0)\right]$$



Step 1: evolve forward in time using PDE to $t_1 = b t_0 \quad b > 1$.

Step 2: rescale $x' = b^{-\phi} x$. Choose ϕ s.t. $\exists$ a fixed pt. $\phi = \frac{1}{2}$ here.

Step 3: rescale u: $u'(0, t_0) = u(0, t_0)$.

$\rightarrow \quad u'(x,t_0) = Ru = Z(b)\, u(b^{\frac{1}{2}} x, b t_0)$

Semi-group: $R_{b_1} * R_{b_2} = R_{b_1 b_2} \Rightarrow Z(b) = b^{y}$ or

$$y = \frac{d \log Z}{\quad}.$$

Fixed pt.: $u^* = R[u^*] \Rightarrow u^*(x,t) = b^{y} u^*(b^{1/2}x, bt)$. $\quad$ all $\log b$

Set $b = \text{const}/t \quad \rightarrow \quad u^*(x,t) = t^{-y} f(x/\sqrt{\kappa t}$

In practice do step 1-> numerically. Very effective numerical scheme. (7)

Here, let's use our renormalized P.T. to do that step:

$$Z(b) = b^{-1/2}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \log b\right] + O(\epsilon^2).$$

$\Rightarrow$ exponent $y = \frac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)$ and

$$u(x,t) = \frac{1}{t^{\frac{1}{2}+\alpha}} f(x/\sqrt{t}) \quad t \to \infty.$$

II. Global Asymptotics.

① Simple linear eqn → boundary layer concept.

$$\epsilon \ddot{y} + \dot{y} + y = 0 \qquad y(0) = 0 \quad y'(0) = 1$$

Naive perturbation theory:

$$y(t) = A(0)e^{-t} - \epsilon t A(0) e^{-t} + O(\epsilon^2), \quad \epsilon t \ll 1.$$

Evolve until δt, $\epsilon \delta t \ll 1$ $\quad y(\delta t) = A(0)(1 - \epsilon\delta t)e^{-\delta t} + O((\epsilon \delta t)^2)$.

Wilson: treat as initial condition and do it again.

$$n \text{ times}, \; n\delta t = t \left\{ \begin{array}{l} y(2\delta t) = y(\delta t)(1 - \epsilon \delta t) e^{-\delta t} + O((\delta t)^2) \\ \vdots \\ y(t) = A(0)(1 - \epsilon \delta t)^{t/\delta t} e^{-t} + \dots \end{array} \right.$$

$$\lim_{\substack{n\to\infty \\ \delta t \to 0 \\ n\delta t = t}} y(t) = A(0) e^{-(1+\epsilon)t}$$

frequency renormalization.

But. Q where's second constant of integration?

$\underline{\underline{A}}$: Boundary layer at $t=0$ of thickness $\delta = O(\epsilon)$

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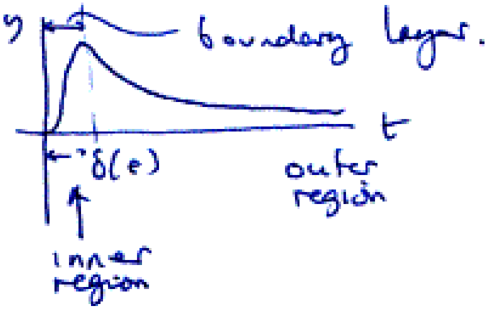
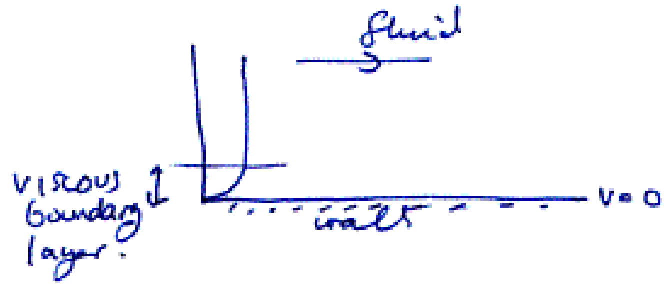
Actual solution looks like

$$y(t) = C_1 e^{-(1+\epsilon)t} + C_2 e^{-\frac{t}{\epsilon} + (1+\epsilon)t} + O(\epsilon^2)$$





(2) Problems with boundary layers example of singular perturbations.
ϵ multiplies highest derivative;

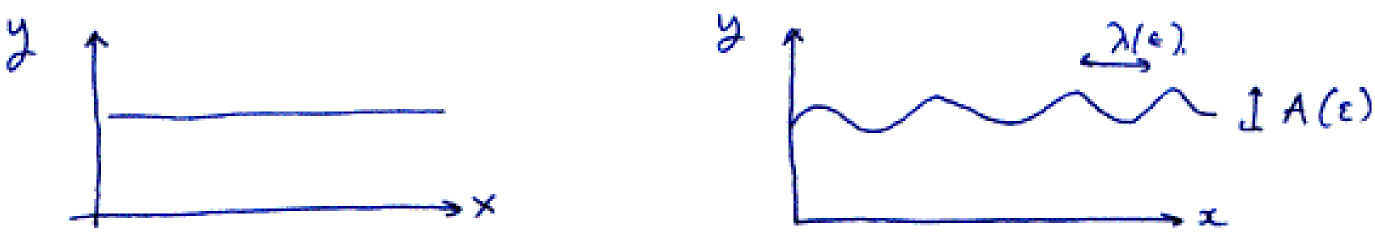
$$[\epsilon = 0] \not\longleftrightarrow [\epsilon \to 0]$$

CM QM.

Other examples: WKB problems

Bifurcation near critical points

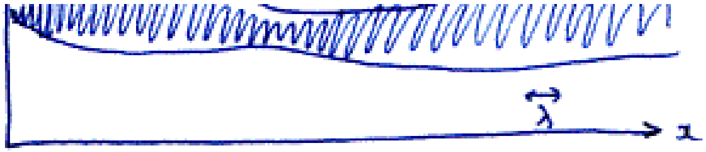
ϵ = control parameter in Rayleigh - Bénard convection



$\epsilon < \epsilon_c$ $\epsilon > \epsilon_c$

Amplitude can vary spatially (e.g.) near walls, topological defects.
$A(x,t;\epsilon)$. What governs?

$A(\epsilon)$ slowly varying on scale



')$\lambda(\epsilon)$.

(9')

Turns out that

$$\frac{\partial A(x,t)}{\partial t} = \nabla^2 A + \epsilon A(1 - 3|A|^2)$$

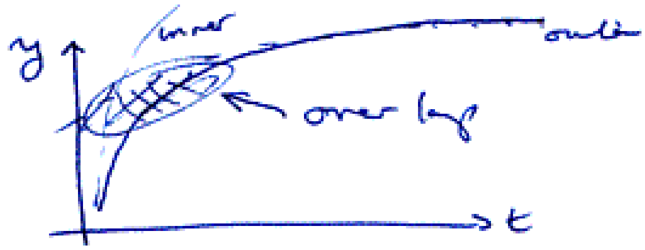
Time dependent Ginzburg-Landau eq. Derived systematically via RG procedure. Universal long wavelength, low frequency behavior governing the PDE near bifurcation point.

(3) What happens when we use RG to solve boundary layers etc?

Conventional: $\epsilon \ddot{y} + \dot{y} + y = 0$. Outer expansion

Rescale $t = \epsilon\tau$ (easy to figure out). $y_{\tau\tau} + y_\tau + y\epsilon = 0$ inner expansion.

Solve inner + outer, find overlapping region of validity in ϵ, t space match constants.



Problems:

(1) Inner + outer expansion naively of form

$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots$

(2) Find that you cannot match unless put in "non-intuitive terms" e.g.

$y_0(t) + \epsilon y_1(t) + \epsilon^{3/2} y_{3/2}(t) + \epsilon^2 y_2(t) \ldots$

or worse.

E.g. $\epsilon y'' + xy' - xy = 0 \quad y(0) = 0 \quad y(1) = e$

— Matching requires and $\epsilon \log \epsilon$ term to be introduced out of nowhere

— RG $\rightarrow$ $y(x) = e^x x^{-\epsilon}\left[1 - \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{\epsilon}}^{\infty} ds\, e^{-s^2/2}\right]$

* just using inner expansion
* just using naive perturbation expansion.
* origin of non-analytic terms in matching is exposed in RG method.

Take home message: RG makes singular perturbation theory mechanical!

(4) How good are the results?

Ex: A 'terrible problem' (switchback).

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \epsilon u \frac{du}{dr} = 0 \qquad u(1) = 0, \quad u(\infty) = 1.$$

RG

$$u(r,\epsilon) = 1 - \frac{e_2(\epsilon r)}{e_2(\epsilon)} + O\left(1/e_2(\epsilon)^2\right)$$

$$e_2(t) = \int_t^\infty dx\, x^{-2} e^{-x} \sim \frac{1}{t} + \ln t + (\gamma - 1) - \frac{t}{2} + \ldots$$

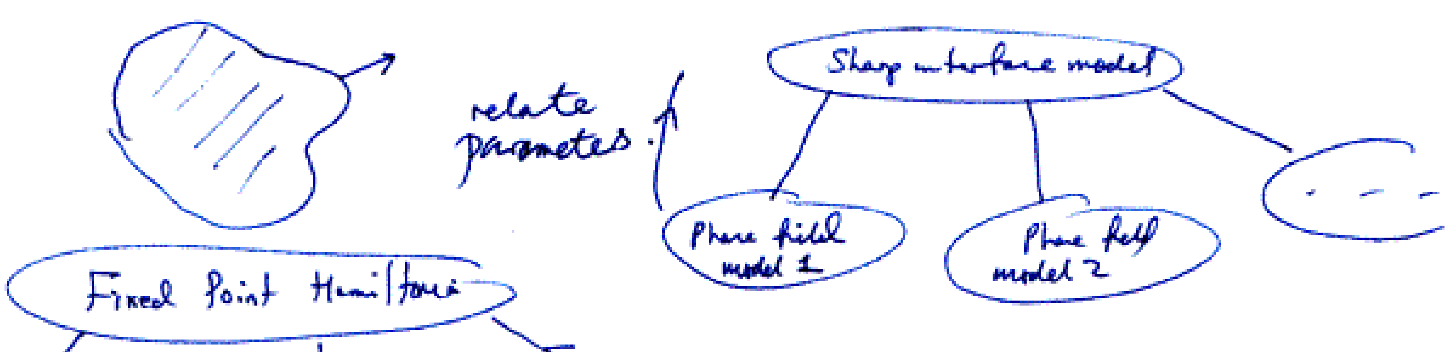
Work well, even when $\epsilon \sim O(1)$.

III. Applications.

(1) Pattern formation problems — convection
spatially extended dynamical system.
e.g. particle beams at Fermilab.

(2) BH collapse (Choptuik phenomenon).

(3) Pattern formation dynamics: snowflakes



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Sharp interface

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$$\partial_t u = D\nabla^2 u \qquad u = \frac{T - T_m}{L/C}$$

$$V_n = -D(\partial_n u_L - \partial_n u_S)$$

$$u_{\text{interface}} = \Delta - d_0(\underline{n})\kappa - \beta(\underline{n})V_n .$$

Phase field.

$$\partial_t u = D\nabla^2 u + \frac{1}{2}\frac{\partial\phi}{\partial t}$$

$$\tau\,\partial_t\phi = W^2\nabla^2\phi - \frac{\delta}{\delta\phi}F(\phi, \lambda u)$$

$$F = \int f(\phi) + \lambda u\, g(\phi)$$

$$f = -\phi^2/2 + \phi^4/4$$

$$\frac{\partial g}{\partial\phi} = (1-\phi^2)^n \qquad n = 1, 2, 3, \ldots$$

Isotropic

Result

$$\lambda = \frac{W_0}{d_0}a_1 \qquad a_1 = \frac{1}{\sqrt{2}} \quad n=1$$

$$\frac{5}{4\sqrt{2}} \quad n=2$$

$$\tau = \frac{W_0^3 a_1 a_2}{d_0 D} \qquad a_2 = 5/6 \quad n=1$$

$$a_2 = \frac{47}{75} \quad n=2$$

$\vdots$

(4) Low Reynolds number flow

Drag on cylinder.

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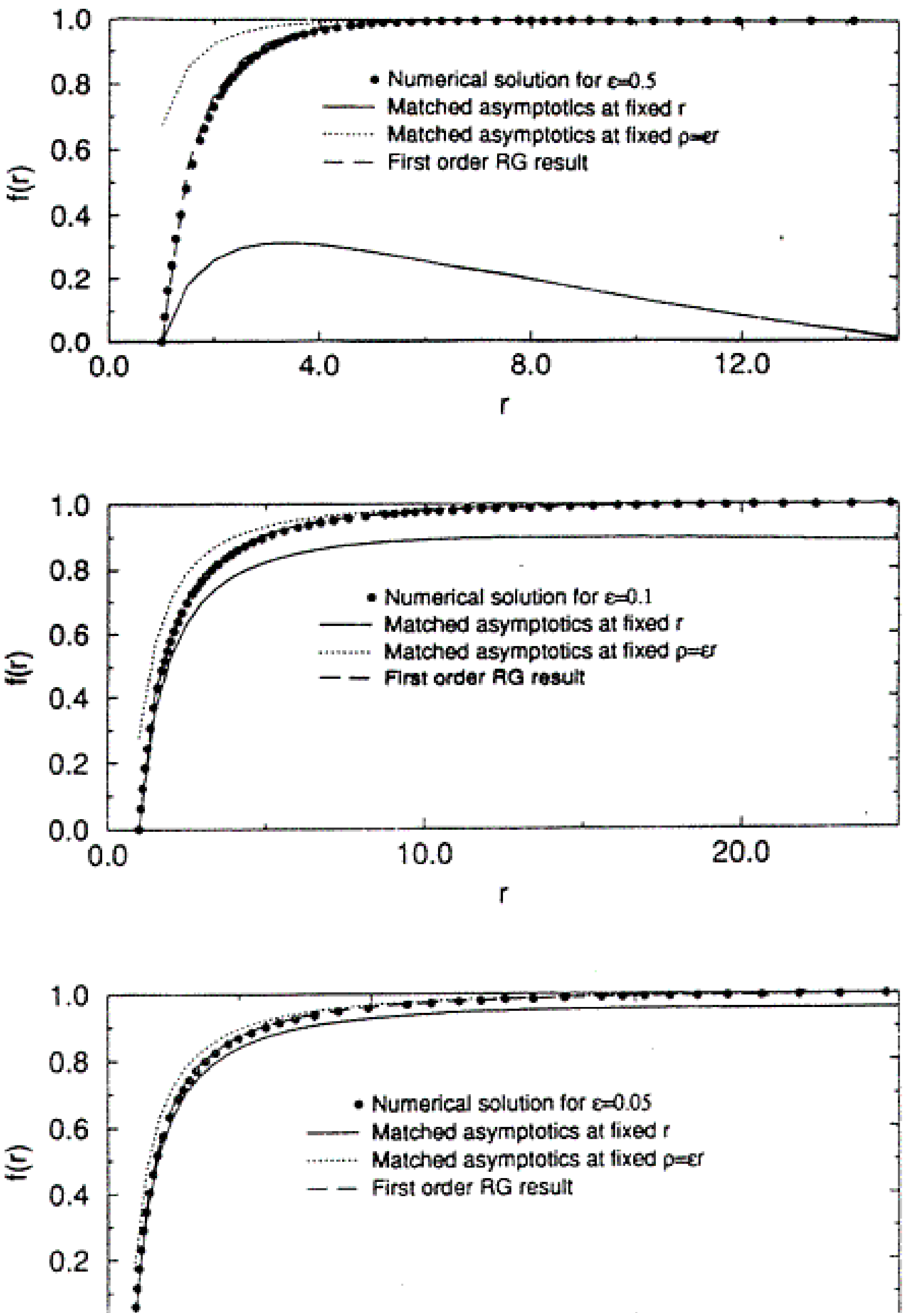
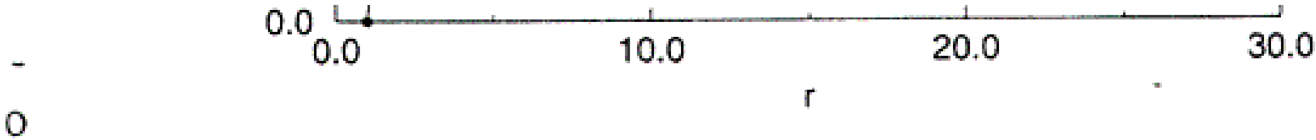

1.0
0.8
0.6
0.4
0.2
0.0
f(r)
0.0
4.0
8.0
12.0
r
• Numerical solution for ε=0.5
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result
1.0
0.8
0.6
0.4
0.2
0.0
f(r)
0.0
10.0
20.0
r
• Numerical solution for ε=0.1
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result
1.0
0.8
0.6
0.4
0.2
f(r)
• Numerical solution for ε=0.05
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result



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