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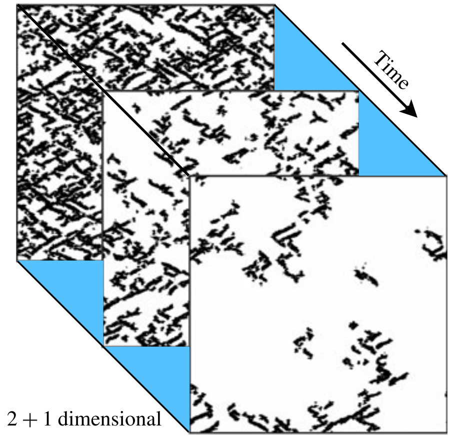
A statistical mechanical phase transition to turbulence in a model shear flow



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It is becoming increasingly clear that the strong spatial and temporal fluctuations observed in a narrow Reynolds number regime around the laminar–turbulent transition in shear flows can best be understood using the concepts and techniques from a seemingly unrelated discipline – statistical mechanics. During the last few years, a consensus has begun to emerge that these phenomena reflect an underlying non-equilibrium phase transition exhibited by a model of interacting particles on a crystalline lattice, directed percolation, that seems very far from fluid mechanics. Now, Chantry *et al.* (*J. Fluid Mech.*, vol. 824, 2017, R1) have developed a truncated-mode computation of a model shear flow, capable of simulating systems far larger and longer than any previous study and have for the first time generated enough statistical data that a high-precision test of theory is feasible. The results broadly confirm the theory, extending the class of flows for which the directed percolation scenario holds and removing any remaining doubts that non-equilibrium statistical mechanical critical phenomena can be exhibited by the Navier–Stokes equations.

Key words: instability, transition to turbulence

1. Introduction

The laminar–turbulence transition in shear flows has presented a challenge to theoretical understanding since Reynolds began the systematic and scientific exploration of the phenomenon in pipe flow, more than 130 years ago (Reynolds 1883). Reynolds used injected dye to follow the flow, and discovered that the onset of turbulence in a pipe does not occur uniformly in space or time. Instead, he found that the dye made flow patterns that were laminar in some regions, but interspersed by local ‘flashes’ of turbulence (nowadays called puffs) where the flow was stochastic and irregular, at least for a certain time τ. Later work showed that as the Reynolds number

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(Re) is increased, puffs split after a time τ_s and occupy a greater fraction of the pipe, until eventually a small disturbance can trigger a growing 'slug' of turbulence that would lead to a completely turbulent flow domain at long times. This sequence of events occurs in a range of Reynolds numbers, approximately $1800 < Re < 2500$, and is very different in character from other well-studied transitions that occur in fluid flow. Thus, from inception, the field of transitional pipe flow turbulence has been preoccupied with a seemingly simple conceptual question: at what Re can the transition to turbulence be said to have taken place?

Modern experiments have confirmed and quantified Reynolds' original findings. Most notably, in a tour de force by Hof *et al.* (2008), Avila *et al.* (2011), it was shown that both the puff lifetime τ and the puff splitting time τ_s increase rapidly with Re (well fit by the functional form $\exp(\exp(a + bRe))$, so that there is no apparent Re which marks the point beyond which a uniform state of turbulence has an infinite lifetime and is the stable state.

These and other characteristics suggest that the laminar–turbulent transition, in pipe flow at least, is sub-critical in nature, with the turbulent state arising discontinuously from the laminar state through a finite amplitude instability as Re exceeds a critical value Re_c whose value is not universal but depends on the perturbation of the flow. Other shear flows, such as plane Couette and Taylor–Couette with a stationary inner cylinder, also transition to turbulence in this way.

Regardless of the specific flow realisation, the natural assumption would be that a description for the transition would emerge from the framework of bifurcation and dynamical systems theory, and indeed, in early work on spatio-temporal intermittency in coupled map lattices, the relevance to transitional pipe turbulence was specifically noted and explored (see Kaneko 1984). In 1986, Pomeau (1986) initiated the modern era of sub-critical transitional turbulence with a prescient comment that brought the topic firmly into the realm of statistical mechanics. He pointed out that the interspersed patches of turbulence in the laminar background near the transition would either grow by diffusing their energy to neighbouring locations, or would die by suddenly, undergoing a fluctuation to the laminar state. From this slender basis, he suggested that the dynamics resembles the behaviour of a statistical mechanical model of particles hopping on a lattice: so-called directed percolation (DP), and he postulated that this model's critical behaviour would describe the sub-critical laminar–turbulent transition. Later theoretical works showed semi-quantitatively that DP behaviour seemed to arise from simple partial differential equation caricatures of spatio-temporal intermittency (see, e.g. Chate & Manneville 1987). Moreover, recent direct numerical simulations (DNS) of pipe flow showed the presence of turbulent energy oscillations between small-scale nascent turbulence and a long-wavelength azimuthal flow (a zonal flow), described by a predator–prey-like stochastic model that could be mapped directly into DP near the critical Re (Shih, Hsieh & Goldenfeld 2016) using statistical mechanics techniques. Finally, recent experiments in a high aspect ratio Taylor–Couette system (Lemoult *et al.* 2016) and in channel flow (Sano & Tamai 2016) show scaling behaviour consistent with the DP predictions. Despite these advances, direct and detailed verification that the Navier–Stokes equations generated stochastic solutions in the DP universality class has seemed out of reach due to the huge system sizes needed to observe the predicted effects. Moreover, an open question remains as to the broad applicability of the DP scenario.

2. Overview

Against this background, Chantry, Tuckerman & Barkley (2017) now report an unprecedented large-scale direct numerical simulation of a planar shear flow, showing in great detail that DP provides a quantitative and detailed description of the complex flow structures that emerge in a large flow domain. The size of the flow domain is very important to capture the necessary phenomena, because of the huge separation of time scales that emerges near the transition. To date, the state-of-the-art was the plane Couette simulations of Duguet, Schlatter & Henningson (2010), conducted in a spatial domain whose $[x, y, z]$ dimensions were $[800h, 2h, 356h]$ where $2h$ is the gap between the walls, and which lasted $O(10^4)$ time units. In contrast, the study of Chantry *et al.* (2017) was in a computational domain of streamwise and spanwise size $2560h \times 2560h$, and lasted $O(10^6)$ time units; thus it was not simply an incremental improvement on the state-of-the-art. In fact, this difference is critically important, because it is well established that the order of a phase transition can be incorrectly ascertained if the domain is too small or the observation time is too short, and Chantry *et al.* (2017) rehearse these arguments using coupled map lattices in the first part of their paper.

So, how did Chantry *et al.* (2017) make such a significant step, and what did they find?

First of all, they chose to study a simplified form of Couette flow between parallel boundaries, known as Waleffe flow. In the simplification, they changed the boundary conditions to be stress free rather than no slip. This trick might seem unphysical but enables researchers to simulate the interior of the flow domain beyond the complicated boundary layer structure at the walls. Secondly, the Waleffe flow, being an outcome of stress-free boundary conditions, has no rapidly varying boundary layers near the walls, and thus can be well represented by only four modes in the direction normal to the walls, while the remaining directions are treated using standard spectral methods.

Turning now to their results, we need first to describe DP in a little more detail. In DP, particles can hop on a periodic lattice in d dimensions, such as square or triangular, but bonds between neighbouring sites are only present with a specific probability p. Particles do not hop if there is no available bond. This percolation process is directed because the paths traced out by the particles in space–time are embedded in a $d+1$-dimensional space, but the trajectory is always moving in the direction of increasing time. The statistical geometry of DP clusters near the critical percolation probability where trajectories first percolate through the system obeys a host of power-law scaling relations, analogous to those arising in the theory of second order or continuous phase transitions. For example, correlation lengths in the space and time dimensions diverge near criticality, a characteristic of second order phase transitions, both in and out of equilibrium (see, e.g. Goldenfeld 1992). Because the correlation length is much larger than the lattice in the critical region, DP behaves essentially like a continuum model, and can describe the Navier–Stokes equations near their critical point. This is precisely analogous to the way in which the liquid–gas phase transition has exactly the same critical behaviour as an array of ferromagnetic electron spins on a crystal lattice (Goldenfeld 1992).

Chantry *et al.* (2017) measured what fraction of the flow was turbulent according to a criterion set in advance for the energy possessed by the velocity deviation from the laminar state. They found that this fraction grows from zero above a critical $Re_c = 173.80$ with a power law $\epsilon^{0.583}$, where $\epsilon \equiv (Re - Re_c)/Re_c$. They also showed that there are patches where laminar and turbulent regions alternate within the flow, with a characteristic length that obeys scaling laws with associated exponents measured, in agreement with the existing phase transition theory for DP. Their work shows beyond reasonable doubt that this simulation based on the Navier–Stokes equations is exhibiting a non-equilibrium phase transition in the DP universality class.

3. Outlook

For the last decade, attention has focused on identifying the statistical features of the laminar–turbulent transition. Now that a consensus is forming that DP is indeed the correct description in several model experimental and computational flows, the next step is surely to understand the mechanism of the transition. Traditionally, this would involve identifying the underlying small-scale flow instability, e.g. a self-sustaining process (see Waleffe 1997). However, this would not be sufficient here. The reason is that a comprehensive description of transitional turbulence needs to account for the statistical mechanical universality class, not just the underlying instability, and this itself is dominated by long-wavelength physics, as we have seen above. This requires an understanding of the nonlinear statistical interactions between the appropriate degrees of freedom, and even in the context of equilibrium phase transition cannot be done systematically starting from a first principles description (e.g. quantum electronics in the case of magnets). Instead, there is no known alternative but to use symmetry, conservation laws and the likely structure of turbulent–mean flow interactions to generate the generic form of an effective long-wavelength description of the transition, as was proposed by Shih *et al.* (2016) for pipe flow. Nevertheless, the combination of a firm understanding of underlying flow instabilities and an equally firm quantitative description of the long-wavelength statistical features of transitional turbulence, as found by Chantry *et al.* (2017), means that the field is now entering an exciting phase, invigorated by the interplay between fluid dynamics and statistical mechanics.

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