Distribution of Maximum Velocities in Avalanches Near the Depinning Transition

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We report exact predictions for universal scaling exponents and scaling functions associated with the distribution of the maximum collective avalanche propagation velocities v_m in the mean field theory of the interface depinning transition. We derive the extreme value distribution $P(v_m|T)$ for the maximum velocities in avalanches of fixed duration T and verify the results by numerical simulation near the critical point. We find that the tail of the distribution of maximum velocity for an arbitrary avalanche duration, v_m, scales as $P(v_m) \sim v_m^{-2}$ for large v_m. These results account for the observed power-law distribution of the maximum amplitudes in acoustic emission experiments of crystal plasticity and are also broadly applicable to other systems in the mean-field interface depinning universality class, ranging from magnets to earthquakes.

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Avalanche phenomena have been observed in a wide variety of disordered systems that exhibit crackling noise near a depinning transition. Examples include Barkhausen noise in soft magnetic materials [1,2], elastic depinning of charge density waves [3,4], dynamics of superconductors [5], seismic activity in earthquakes [6], acoustic emission in mesoscopic crystal plasticity [7], and fracture propagation [8]. Although these varying materials or systems have different microscopic details, on long length scales the statistical scaling behavior of avalanches appears to be universal. For example, the distributions of avalanche sizes in sheared crystals and in slowly magnetized soft magnets are both captured by the mean field theory of a slowly driven elastic interface in a disordered medium [4,6,9–12].

Recent experimental studies of slip avalanches in mesocopic crystal plasticity have reported that the distribution of the maximum amplitude A_m of the acoustic emission (AE) signal from each avalanche follows a power law $P(A_m) \sim A_m^{-\mu}$, where the exponent $\mu \approx 2$ [7,13–17]. Since each avalanche contributes with only one maximum amplitude to the histogram, many events are required to obtain good statistics for $P(A_m)$. Thus, the variations in the experimental values of μ depend on the experimental statistics. Owing to the proportionality between the AE amplitude A_m and the collective velocity v_m of dislocations [14], the distributions $P(A_m)$ and $P(v_m)$ should be characterized by the same scaling exponents and scaling functions. So far, a theoretical prediction for the value of the exponent μ has been lacking.

In this Letter, we present the first theoretical calculation of the maximum velocity distribution, establishing a connection to the known classes of extreme value statistics (EVS) of correlated variables. In particular, we derive the distribution of maximum velocities $P(v_m)$ from a mean field interface depinning model. We first show that the probability distribution function (PDF) of the maximum velocity for avalanches of fixed duration T follows a universal scaling form $P(v_m|T) = (2v_mT)^{-1/2}F(\sqrt{2v_m/T})$, with a scaling function $F(x)$ that can be derived exactly by a mapping to an equivalent problem of random excursions of Brownian motion in a logarithmic potential. Although a general theory of extremal statistics for strongly correlated variables is not known, much progress has been made already for several classes of power-law correlated noise with an $1/\omega^\alpha$ (where ω is the frequency) power spectrum. Brownian noise corresponds to the particular case where $\alpha = 2$ [18,19]. The EVS of power-law correlated noise typically have a robust scaling form, but the scaling function depends on boundary conditions, the value from which the maximum is measured, as well as other constraints on the time evolution. For example, different scaling functions are obtained for the maximum heights of periodic Gaussian interfaces: if the maximum is measured relative to the spatially averaged height, the corresponding EVS is determined by the so-called Airy distribution function [18,20–22], whereas measuring the maximum relative to the boundary value leads to the Rayleigh distribution [19,23]. Here we demonstrate that our problem of maximum heights of amplitudes of mean field avalanches is equivalent to a related problem whose exact solution obeys the same scaling form with a distinct function. Finally, we show that the overall distribution scales like $P(v_m) \sim v_m^{-2}$ by integrating $P(v_m|T)$ against the duration PDF $F(T)$. The results of this study are expected to be broadly applicable to plasticity, earthquakes, Barkhausen noise in soft magnets, and many other systems in the mean field interface depinning universality class.

Our starting point is a zero-dimensional model of a slowly driven elastic interface in a disordered medium, also known as the Alessandro-Beatrice-Bertotti-Montorsi

(ABBM) model [24], corresponding to the dynamics of a particle pulled by an elastic spring and an external field through a random force landscape. The position of the particle $u(t)$ corresponds to the center-of-mass displacement of the interface $u(t) = L^{-d} \int d^d x u(\boldsymbol{x}, t)$, given the local displacement $u(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ along the interface length L and time t for an interface of dimension d embedded in a $(d+1)$-dimensional space. In the ABBM model, the evolution of the particle velocity $v = du/dt$ is obtained by a time-differentiation of the mean field equation of motion of the interface and given as [10,24,25]

$$\frac{dv}{dt} = -kv + c + \sqrt{v}\xi(t), \tag{1}$$

where c is the constant rate, k is the elastic coupling constant, and $\xi(t)$ is Gaussian white noise with autocorrelation $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$ where D is a constant measure of disorder. In numerous studies, this model has been shown to reproduce well the universal scaling laws for the size and duration distributions near quasistatic depinning for systems with long-range interactions [1,10,24–26]. It is relevant for the calculations below to recall the power law decay of the distribution of avalanche durations $F_T(T) \approx T^{-(2-c/D)} f_T(\frac{T}{T^*})$, with a rate-dependent exponent [1,24,25] and such that, for $T \ll T^*$ and $c \to 0$, the distribution of durations follows the mean field scaling law $F_T(T) \approx T^{-2}$. The exponential scaling function $f_T(T/T^*)$ and the cutoff $T^*(k)$ to the scale invariance can be computed analytically in the limit of $c = 0$ [25,27]. The probability distribution for the velocity follows the Fokker-Planck equation

$$\partial_t P(v,t) = \partial_v((kv - c)P(v,t)) + D\partial_v^2(vP(v,t)), \tag{2}$$

which has a steady state solution given by

$$P(v) = v^{-1+\tilde{c}} \frac{\tilde{k}^{\tilde{c}}}{\Gamma(\tilde{c})} e^{-\tilde{k}v}, \tag{3}$$

where $\Gamma(z)$ is the Gamma function, $\tilde{c} = c/D$, and $\tilde{k} = k/D$ [1,24,25,27]. The power-law exponent depends linearly on the driving rate c, such that in the adiabatic limit $c \to 0$, the distribution approaches the well-known v^{-1} scaling, which has been verified by experiments on the dynamics of domain walls in ferromagnets [28].

Maximum velocity distribution for avalanches of fixed duration.—By a change of variables to $x = 2\sqrt{v}$, Eq. (1) transforms to an additive-noise Langevin equation $dx/dt = -kx/2 + (2c - D)/x + \xi(t)$. The additional $1/x$ term comes from the Ito interpretation of the multiplicative noise in Eq. (1). This choice yields the correct Eq. (2). Thus, in the adiabatic limit, near depinning, where $c \to 0$ and $k \to 0$, the velocity evolution can be mapped onto a one-dimensional (1D) Brownian motion in a logarithmic potential. An avalanche of duration T corresponds to an excursion, i.e., a path $x(t)$ with $x(0) = x(T) = 0$ and $x(t) > 0$ for $0 < t < T$. The extreme displacement distribution for Brownian excursions can be derived using the path integral formalism found in Refs. [21,23]. We adapt this method to our problem and determine the cumulative distribution $C_{RW}(x_m|T)$ of the maximum displacement during excursions for a Brownian motion in a logarithmic trap. The cumulative distribution can be defined as $C_{RW}(x_m|T) =$

$$\lim_{\epsilon\to 0} \frac{\int_{x(0)=\epsilon}^{x(T)=\epsilon} \mathcal{D}x e^{-\int_0^T dt \mathcal{L}_E} \prod_t \Theta(x(t))\Theta(x_m - x(t))}{\int_{x(0)=\epsilon}^{x(T)=\epsilon} \mathcal{D}x e^{-\int_0^T dt \mathcal{L}_E} \prod_t \Theta(x(t))}, \tag{4}$$

where the Lagrangian is given by $\mathcal{L}_E = \frac{1}{4D}(\dot{x} + \frac{1}{x})^2$. The theta function products in the numerator indicate that only paths that stay positive-valued between $t = 0$ and $t = T$ and have a maximum distance from the origin not greater than x_m are counted. The denominator is a normalization factor, counting any excursion of duration T without regard to its maximum value. The Fokker-Planck equation [Eq. (2)] with $c = k = 0$ in terms of the variable x is Bessel's equation of order 1; thus, the path integrals from Eq. (4) can be written as the matrix elements $\langle \epsilon | \exp(-\hat{H}T)|\epsilon\rangle$ of the Hamiltonian $\hat{H} = -\partial_x^2 - \partial_x/x + 1/x^2$ with appropriate boundary conditions and then expanded in terms of Bessel functions (details are presented in Ref. [29]). From $C_{RW}(x_m|T)$, the PDF $P(x_m|T) = \partial_{x_m} C(x_m|T)$ is determined. We find that the $P(x_m|T)$ has the scaling form

$$P(x_m|T) = \frac{1}{\sqrt{2DT}} F\left(\frac{x_m}{\sqrt{2DT}}\right), \tag{5}$$

with scaling function

$$F(x) = \frac{1}{x^5} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(J_2(\lambda_n))^2} \left[\frac{\lambda_n^2}{x^2} - 4\right] e^{-\lambda_n^2/2x^2}, \tag{6}$$

where λ_n is the nth zero of the Bessel function $J_1(x)$. From Eq. (5), it also follows that the average maximum displacement scales with duration as $T^{1/2}$, like the average maximum relative heights of fluctuating interfaces [20,21,23]. Returning to the physical variable v, we find that the maximum velocity distribution in avalanches of fixed duration T has the scaling form

$$P(v_m|T) = \frac{1}{\sqrt{2v_m DT}} F\left(\sqrt{\frac{2v_m}{DT}}\right). \tag{7}$$

The average maximum velocity dependence on avalanche duration T can be obtained as the first moment of the conditional distribution

$$\langle v_m|T\rangle = \int_0^\infty dv_m v_m P(v_m|T) = DT, \tag{8}$$

where we used the fact that $\int dx x^2 F(x) = 2$.

Using the same method, we also determine that the PDF of the instantaneous velocity v at time t in an avalanche of duration T is given by

$$P(v, t|T) = v\left(\frac{T}{Dt(T-t)}\right)^2 e^{-vT/(Dt(T-t))}. \tag{9}$$

The first moment of $P(v, t|T)$ gives $\langle v(t)|T\rangle = 2Dt(T-t)/T$, which is the parabolic average avalanche shape discussed in Refs. [2,30,31].

We have verified Eqs. (7) and (8) numerically by integrating Eq. (1) (see Fig. 1). Large durations ($T \sim 1000$) must be explored for the scaling function to converge to the one predicted by our continuum derivation, but the results are in accord with predictions. We obtained improved statistical results for data collapse to the same scaling function using the computationally efficient discrete velocity shell model [6,31], which obtains its scaling regime at smaller durations. These results will be reported elsewhere [29].

Although our analytical calculation was performed exactly only for $k = c = 0$, the scaling form in Eq. (7) gives a good collapse of simulation data away from criticality as well, even when including durations $T \sim T^*(k)$ [29]. Therefore the dependence of the scaling form on k is likely to be weak. Since the driving rate parameter $\tilde{c}$ is dimensionless, one might anticipate that nonzero values of the driving rate c modify the scaling function but not the scaling form. Indeed, the exact form of the modification can be calculated analytically with a slight generalization of the above calculation [29].

Maximum and instantaneous velocity statistics.—We now investigate the maximum velocity distribution integrated over all durations. This distribution is equivalent to the $P(A_m)$ of the maximum AE amplitude, A_m, deduced from the time series in AE experiments on crystal plasticity. From Eq. (9), we can determine the PDF of the maximum avalanche velocity $P(v_m)$ by integrating $P(v_m|T)$ over avalanche durations T weighted by their distribution $F_T(T) \sim T^{-2+\tilde{c}}$, for $T \ll T^*$. Our numerics indicate that $P(v_m|T)$ satisfies Eq. (7) at least for durations $T < T^*$. Thus, the distribution of $P(v_m)$ is

$$P(v_m) \sim \int_0^{T^*} \frac{dT}{T^{2-\tilde{c}}} P(v_m|T), \tag{10}$$

and near depinning, where $T^* \to \infty$, we have

$$P(v_m) \sim v_m^{-2+\tilde{c}}. \tag{11}$$

Similarly, we obtain the PDF $P(v)$ for the instantaneous velocity v at an arbitrary time by integrating $P(v, t|T)$ over the time spent in an avalanche of duration T and then over the distribution of durations, giving

$$P(v) = \int_0^{T^*} dT \int_0^T dt P(v, t|T)F(T) \tag{12}$$

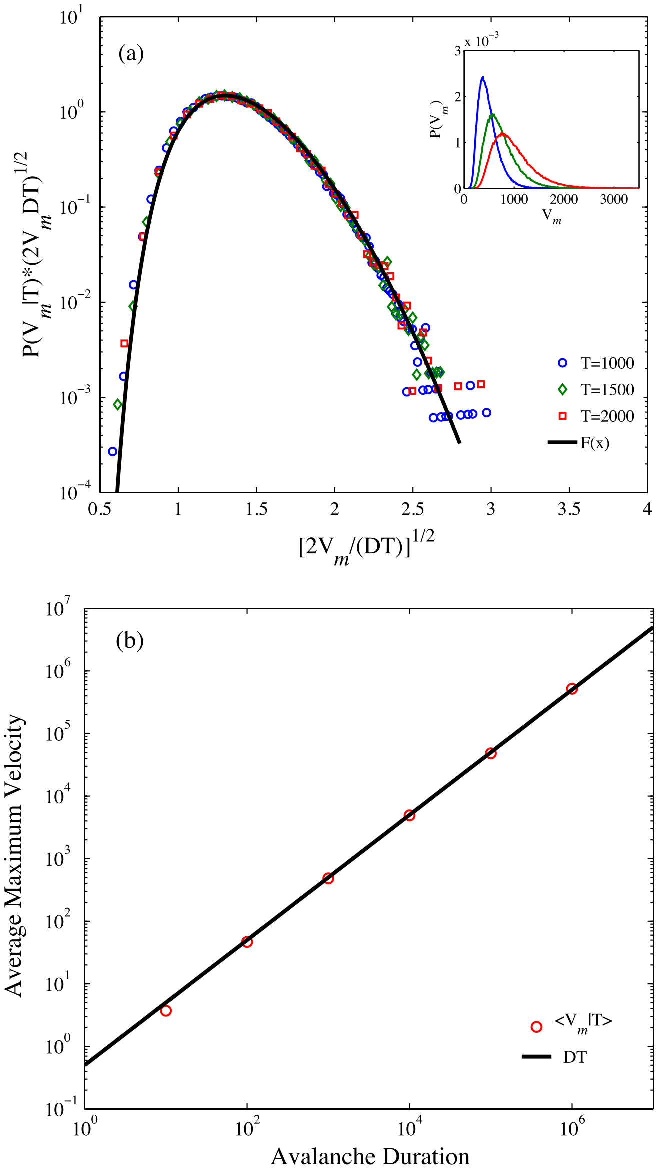

FIG. 1 (color online). (a) Data collapse of the PDF $P(v_m|T)$ from numerical integration of Eq. (1) in the Ito interpretation with parameter values $k = c = 0$ and $D = 1/2$. Large durations ($T \sim 1000$) are required to obtain the scaling regime where Eq. (7) holds. The collapse fits very well with the analytically determined $F(x)$, which is represented by the solid line. The inset figure shows the $P(v_m|T)$'s for different durations before the rescaling. In panel (b), we show $\langle v_m|T\rangle$ as a function of T, with the solid line representing the analytical solution from Eq. (8) with $D = 1/2$.

$$\sim v \int_0^{T*} \frac{dT}{T^{3-\tilde{c}}} G(v/DT), \tag{13}$$

where $G(x) = \int_0^1 du(u(1-u))^{-2} \exp[-x(u(1-u))^{-1}]$. $G(x) \sim x^{-1}$ for $x \ll 1$ and decays exponentially for $x \gg 1$, so in the limit $T^* \to \infty$, we recover the $P(v) \sim v^{-1+\tilde{c}}$ scaling predicted by the steady state equation for $\tilde{c} < 1$ [10,24,25,27]. In Fig. 2, we show numerically calculated

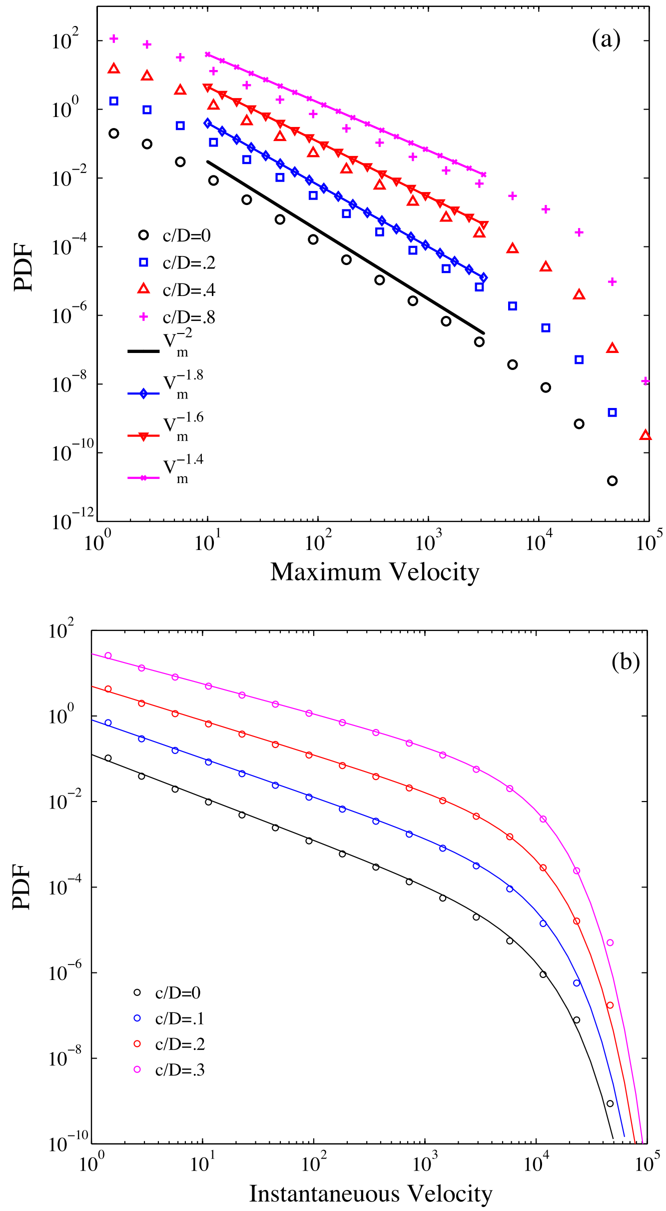

FIG. 2 (color online). In panel (a), the PDF $P(v_m)$ obtained from numerical integration of Eq. (1) is shown for several values of $\tilde{c}$, all with $k = 10^{-4}$ and $D = 1/2$. The PDFs are offset vertically so they can be clearly distinguished. Above each curve, a guideline is drawn indicating the power law analytically predicted from Eq. (11). In panel (b), we show the instantaneous velocity PDF $P(v)$ for several values of $\tilde{c}$. The solid lines represent the functional form predicted by Eq. (3). Again, the PDFs are offset for visibility.

PDFs $P(v_m)$ and $P(v)$ for various values of c. The distributions agree with the predictions of Eqs. (11) and (3).

In addition to the exponents, it would be interesting to measure the predicted scaling form of the $P(v_m|T)$ over fixed durations from AE experiments, in the corresponding regime where the distribution of maximum amplitudes $P(A_m) \sim A_m^{-2}$ was observed [7,13–17]. The distribution $P(v_m|T)$ was calculated exactly only at the depinning transition with $k = c = 0$, but numerical evidence strongly suggests that an indistinguishable scaling form occurs away from the transition, whose dependence on the elastic coupling constant and the driving rate still needs to be studied in more detail. Finally, what happens beyond mean field theory remains an open question, despite the apparently good agreement of our calculations with available experimental data. For instance, it is unclear to us whether the exponent μ in $P(v_m) = v_m^{-\mu}$ is expressible in terms of the other avalanche exponents (τ, $\sigma \nu z$, etc.) or if it is independent.

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