PHYSICAL REVIEW E **79**, 065306(R) (2009)

Friction factor of two-dimensional rough-boundary turbulent soap film flows

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(Received 11 August 2008; revised manuscript received 29 May 2009; published 25 June 2009)

We use momentum-transfer arguments to predict the friction factor f in two-dimensional turbulent soap film flows with rough boundaries (an analog of three-dimensional pipe flow) as a function of Reynolds number Re and roughness r, considering separately the inverse energy cascade and the forward enstrophy cascade. At intermediate Re, we predict a Blasius-like friction factor scaling of $f \propto \mathrm{Re}^{-1/2}$ in flows dominated by the enstrophy cascade, distinct from the energy cascade scaling of $\mathrm{Re}^{-1/4}$. For large Re, $f \sim r$ in the enstrophy-dominated case. We use conformal map techniques to perform direct numerical simulations that are in satisfactory agreement with theory and exhibit data collapse scaling of roughness-induced criticality, previously shown to arise in the three-dimensional pipe data of Nikuradse.

DOI: 10.1103/PhysRevE.79.065306 PACS number(s): 47.27.ek, 47.27.nf

Turbulent flows are marked by rich structure over a range of scales—they host fluctuations, vortices, tangles, and other coherent structures that continue to defy a detailed analytical understanding [1,2]. When parametrized in terms of the typical flow speed U, characteristic length scale L and kinematic viscosity of the fluid ν three-dimensional turbulence exhibits universal phenomena as the Reynolds number $\mathrm{Re} \equiv UL/\nu \to \infty$. Most famously, in a theory referred to as K41 [3,4], the dependence of the fluctuation energy spectrum $E(k)$ on the wave number and mean energy-transfer rate $\bar{\epsilon}$ occurs in a way that is independent of ν: $E(k) = \bar{\epsilon}^{2/3} k^{-5/3}$ for values of the wave number in the so-called inertial range, intermediate between the scales of forcing and the scales where molecular viscosity becomes significant. In this inertial range, turbulent eddies break up into smaller eddies through a mechanism, which is to a first approximation Hamiltonian, and results in a cascade of energy to smaller length scales [5].

During the 1930s, Nikuradse undertook a systematic series of measurements of the pressure drop across a turbulent pipe flow as a function of Re [6] and also as a function of r/D (the scale of the roughness of the pipe walls r) normalized by the pipe diameter D [7]. The former measurements provided strong support for the turbulent boundary layer concept [8] and have been replicated and surpassed only recently [9], while the latter measurements, despite recent efforts [10,11], remain to this day the most complete data set of its kind, spanning 3 orders of magnitude in Reynolds number and a decade in the dimensionless roughness r. These data reveal that the frictional drag experienced by a turbulent fluid in a pipe with rough walls is a nonmonotonic and complicated function of Reynolds number and roughness, which despite intense interest and practical importance (see, e.g., Ref. [8]), has only begun to be understood [12,13] through two related developments.

First, Gioia and Chakraborty [12] estimated the momentum transfer between the walls of the pipe and the flow, explicitly taking into account the presence of roughness. Their resultant formula for the dimensionless friction factor $f = \frac{\partial_x P D}{2\rho U^2}$ is expressed in terms of the turbulent kinetic-energy spectrum $E(k)$ and, thus, makes a direct connection between a macroscopic flow property and the velocity field correlations. Second, Goldenfeld [13] pointed out that the power-law behavior of Nikuradse's friction factor data in the regimes $\mathrm{Re} \to \infty$ and $r/D \to 0$ was analogous to critical phenomena, where the inverse Reynolds number and roughness play similar roles to, for example, the coupling constant and external magnetic field in an Ising model. Consequently, the dependence of Nikuradse's data on Re and r can be collapsed onto a universal function with sufficient precision for intermittency corrections to be extracted [14]. These results show that the friction factor reflects the nature of the turbulent state through its dependence on the energy spectrum and that the turbulent state is itself a manifestation of a nonequilibrium critical point at $\mathrm{Re} = \infty$ and $r/D \to 0$.

The purpose of this Rapid Communication is to test the claims of Refs. [12,13] in a context where detailed calculations are in principle possible: the case of two-dimensional (2D) soap film turbulence [15,16]. Here, a soap film is supported between two vertical wires, and the draining flow provides a versatile laboratory for exploring two-dimensional turbulence [16]. It is well understood that the nature of turbulence in 2D is different from three-dimensional: there is no vortex stretching, for example. Nevertheless, turbulent phenomena exist and possess the novelty that there are two cascades: an energy inverse cascade that runs from small to large scales [17,18] and a forward cascade [18] in the enstrophy $\Omega \equiv |\nabla \times \mathbf{v}|^2$, where $\mathbf{v}$ is the fluid velocity field. This enstrophy cascade yields an energy spectrum $E(k) = \beta^{2/3} k^{-3}$, where β is the rate of transfer of enstrophy.

Prior work, dating back to Prandtl and others (for a review, see Ref. [8]) is not able to make a prediction about the friction factor in these cases because it has no specific representation of the nature of the turbulent state and, in particular, is disconnected from the energy spectrum. On the other hand, the momentum-transfer theory of Gioia and Chakraborty [12] can reflect the character of 2D turbulent states, as expressed by the energy spectrum, through the dependence of the friction factor on Re and r. We show below that the momentum-transfer theory predicts a significant dependence of the friction factor on the nature of the turbulent cascade, one that we observe in direct numerical simulations reported here, and which obeys the scaling predicted by roughness-induced criticality. Thus, our direct numerical calculations agree well with the momentum-transfer and roughness-induced criticality picture and strongly suggest that the standard picture of turbulent boundary layers is incomplete.

Calculation of the friction factor scaling laws in 2D. In the momentum-transfer theory of Gioia and Chakraborty, the friction factor is shown to be proportional to the root-mean-square velocity fluctuation u_s at a scale s determined by the larger of the roughness r or the Kolmogorov scale η_K. Since $E(k)dk$ represents the turbulent kinetic energy in the wavenumber band between k and $k+dk$, it follows that

$$u_s = \left[\int_{1/s}^{\infty} E(k)dk \right]^{1/2}. \tag{1}$$

Anisotropy near the wall has only a small effect [19] on the low-order structure function used in our calculation. For simplicity, using the K41 form for $E(k)$, we obtain $f \propto \bar{\epsilon}^{1/3} s^{1/3}$. With the limiting forms for s at large and small Reynolds number, we obtain the predictions of the empirically observed Blasius regime[20], in which the friction scales as $\mathrm{Re}^{-1/4}$ for small but turbulent Reynolds numbers, and the Strickler regime [21] at large enough Re, where the friction factor is independent of Re and only depends on the roughness through the relation $f \propto (r/D)^{1/3}$.

In two-dimensional turbulent systems, both the energy cascade and the enstrophy cascade may be observed or they may occur individually [22] depending on the manner of energy injection and the scale at which it occurs. The two-dimensional inverse-cascade friction factor is the same as the case of three-dimensional flows, with a Blasius scaling of $f \propto \mathrm{Re}^{-1/4}$ and a Strickler scaling of $f \propto (r/D)^{1/3}$. The energy spectrum due to the enstrophy cascade leads to a new prediction for the friction factor: a scaling of $f \propto \mathrm{Re}^{-1/2}$ in the Blasius regime and $f \propto (r/D)$ in the Strickler regime. These are our central predictions, which we seek to verify by numerical simulation in the next section. In general, the friction factor corresponding to any conserved quantity (such as helicity) with units $[\mathrm{L}]^a[\mathrm{T}]^b$ is $f \propto \mathrm{Re}^{-(1-\phi)/(2-\phi)}$ (Blasius regime) and $f \propto (r/D)^{1-\phi}$ (Strickler regime), where $\phi \equiv a/(1-b)$.

Simulations of 2D turbulent rough-pipe flows. To test the momentum-transfer theory's prediction of the friction factor in 2D, we have performed simulations for a range of Reynolds numbers and single-wavelength roughness, both with grid-generated turbulence and turbulence generated by wall roughness. The roughness of the wall breaks translational invariance and means that one cannot simply solve the Navier-Stokes equations using spectral methods. We have overcome this difficulty by using a judiciously chosen conformal map technique, allowing us to use a spectral method to satisfy incompressibility. The SMART algorithm [23] is used to calculate the advection of the velocity field. The friction factor is measured by computing the pressure drop necessary to maintain the average flow velocity over the periodic domain.

To simulate a rough-walled pipe, we apply a conformal map of the form $w = z + r\exp(ikz)$, where the aspect ratio is held constant $(rk=3/4)$ and the wave number k may be varied to produce roughness of different scales. Note that r plays the role of roughness in Nikuradse's experiments, but our aspect ratio is 3/4 and not unity as in his experiments. This conformal map results in the addition of two-body force terms to the Navier-Stokes equation in the transformed (rectangular) domain, in addition to an overall weighting factor deriving from the changed volume of each cell,

$$|g'|^2 \frac{\partial \overline{\mathbf{V}}}{\partial t} + (\overline{\mathbf{V}} \cdot \boldsymbol{\nabla})\overline{\mathbf{V}} = \nu \nabla^2 \overline{\mathbf{V}} + \frac{|\overline{\mathbf{V}}|^2}{|g'|^2}\mathbf{A} + \frac{2\nu}{|g'|^2}\mathbf{A}^{\perp}(\boldsymbol{\nabla} \times \overline{\mathbf{V}}) \tag{2}$$

Here $|g'|^2 = x_u^2 + x_v^2 = y_u^2 + y_v^2 = x_u y_v - x_v y_u$, $\overline{\mathbf{V}}$ is the velocity in the transformed coordinates, and the vector $\mathbf{A}$ is defined as

$$\mathbf{A} \equiv \begin{bmatrix} x_u y_{uv} + x_v x_{uv} \\ x_u x_{uv} - x_v y_{uv} \end{bmatrix}. \tag{3}$$

We use a simulation domain of 2048×512 to simulate a section of pipe of diameter 1 and length 4. After initializing the velocity field, we allow the system to evolve for a sufficient number of pipe transits so that the system is fully turbulent (one pipe transit corresponds to four units of time as the mean flow velocity is set to 1 in the simulation units). The smaller the roughness, the more transits are needed. This results in roughness-generated turbulence, in which case the observed energy spectrum is dominated by the inverse cascade, as shown in Fig. 1.

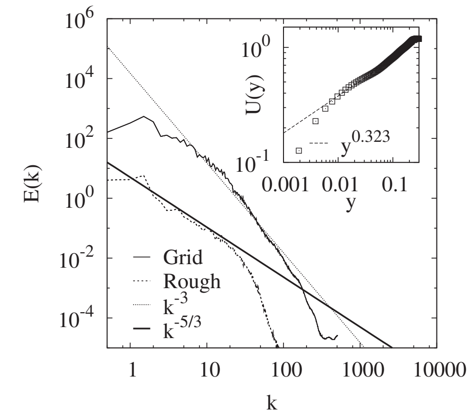

FIG. 1. Energy spectra for grid- and roughness-generated turbulence. Grid-generated turbulence exhibits the k^{-3} enstrophy cascade, whereas roughness-generated turbulence exhibits the $k^{-5/3}$ inverse-cascade scaling. Inset: simulated wall velocity profile of grid-generated turbulence in a smooth pipe at Re=60 000. The profile is consistent with a power law with exponent 0.323 ± 0.005. We predict an exponent of 1/3 for enstrophy-cascade turbulence.

In order to attain an enstrophy-dominated flow, we used a technique suggested by the observations reported by Rutgers [22]. We simulated grid-generated turbulence by placing a series of cylinders at the mouth of the pipe; in each cylinder, we set the velocity field to zero every time step. After one pipe transit, the velocity field is fully developed. We then remove the grid and allow the turbulence to decay for a

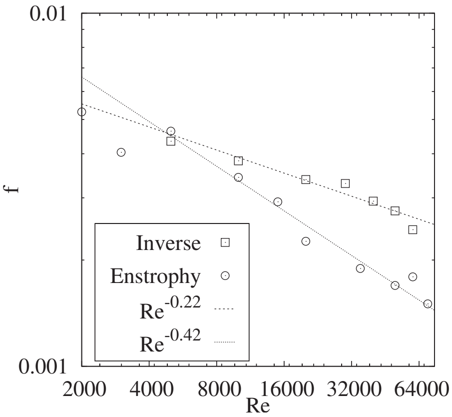

FIG. 2. Scaling of the friction factor with respect to Re for inverse-cascade and enstrophy-cascade dominated flows in 2D. The roughness is $r/D=0.067$, and the data have been averaged over a time of five pipe transits.

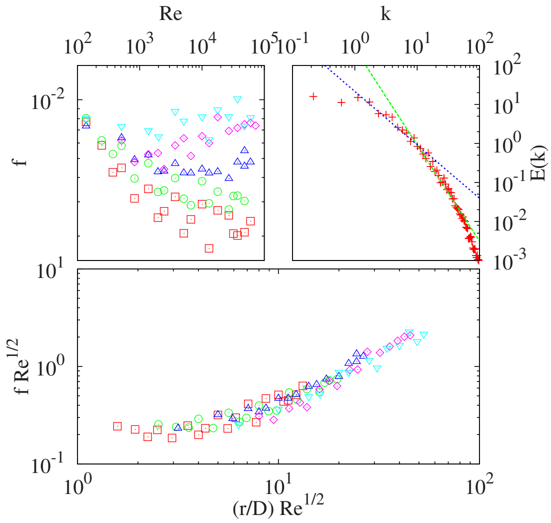

FIG. 3. (Color online) The bottom inset shows the enstrophy-cascade data collapse of the friction factor curves for nondimensional roughness 0.05 (○), 0.08 (□), 0.1 (△), 0.16 (◇), and 0.2 (▽) over a range of Reynolds numbers from 1000 to 80000. The top left inset shows the unscaled friction factor data. The top right inset shows the energy spectrum at $r/D=0.08$ and $Re=80\,000$. The straight lines correspond to $k^{-5/3}$ and k^{-3}.

transit before we begin to measure the friction factor and other flow properties. We have observed energy spectra dominated by the enstrophy cascade in this system, as shown in Fig. 1.

Our simulation results at small values of the dimensionless roughness ($r/D=0.067$) are plotted in Fig. 2. These results were obtained by averaging over five full pipe transits, yielding reproducible values for the friction factor, with controlled error bars, as shown. For this flow, we observe an approximate power-law scaling of the friction factor with Reynolds number, with an exponent -0.22 ± 0.03 together with an energy spectrum dominated by the inverse cascade. In the case of grid-generated decaying turbulence, corresponding to an enstrophy-cascade dominated spectrum, we observe an exponent of -0.42 ± 0.05. These results are within satisfactory agreement with the scalings of $-1/4$ and $-1/2$, respectively, predicted for the 2D Blasius regime on the basis of a momentum-transfer argument.

We cannot reach sufficiently high Reynolds numbers to observe a pure Strickler regime, but we can verify the Strickler scaling exponent with data collapse. In three dimensions or in a system dominated by the inverse cascade, we expect data collapse when plotting $f\mathrm{Re}^{1/4}$ against $(r/D)\mathrm{Re}^{3/4}$ [13]. For the enstrophy cascade, these variables should be $f\mathrm{Re}^{1/2}$ and $(r/D)\mathrm{Re}^{1/2}$, respectively. We have observed previously that in the presence of roughness, the spectrum is dominated by the inverse cascade. However, we have found that by adding a small amount of random forcing to the velocity field, the enstrophy cascade may be observed even in a rough pipe though it may be coexistant with an inverse cascade. Using this method, we can obtain the roughness dependence of the friction factor in an enstrophy-cascade dominated flow. The collapse of the friction factor curves using the enstrophy-cascade variables is shown in Fig. 3. The collapse is quite good, despite an apparent shallowness to the Blasius regime in the raw data. This shallowness is likely caused by the presence of a small amount of roughness, modifying the expected $\mathrm{Re}^{-1/2}$ scaling at larger Reynolds numbers. We have neglected intermittency, which is negligible in 2D [24].

Relationship of the friction factor to the velocity profile. Following Prandtl [25], we have calculated the mean velocity profile $u(y)$ as a function of distance from a wall y and for the enstrophy cascade this yields $u(y)\sim y^{\alpha}$ with $\alpha=1/3$, corresponding to the Blasius regime. For a general conserved quantity, $\alpha=(1-\phi)/(3-\phi)$. This relation depends on the zero roughness limit. In [26], it has been shown that roughness modifies the velocity profile so as to increase the apparent scaling exponent. Other work [27,28] also considers the influence of rough walls on the velocity profile and near-wall scaling.

In our simulations of smooth-pipe enstrophy-cascade turbulence, we have measured the velocity profile and found the power-law scaling exponent $\alpha=0.323\pm0.005$ between $0.01D$ and $0.1D$, as shown in the inset of Fig. 1, close to the predicted $\alpha=1/3$. In the case of our rough-pipe simulations, the velocity profile yielded an exponent of 0.333 ± 0.002, significantly steeper than the predicted $\alpha=1/7$ that applies in the smooth inverse-cascade case. Our interpretation is that this is due to the spectral contamination from an enstrophy cascade, as in the case of the simulations with random forcing that we presented. The momentum-transfer theory integral has an upper limit that is comparable with the Kolmogorov lengthscale at low roughness, and so in that case the small-k part of the energy spectrum controls the friction factor scaling. Because of this, we would expect to see a velocity profile consistent with the enstrophy cascade until the roughness or Reynolds number was high enough to place the crossover between the inverse cascade and contaminant enstrophy cascade below the scale of the roughness.

Our results for the power-law Blasius regime in a 2D enstrophy-dominated turbulence show convincingly that this regime is more than an empirical fit and has a dynamical significance. Our direct numerical simulations support the fundamental connection between spectral structure and friction factor scaling, which is manifested in the observed roughness-induced criticality.

We are grateful for valuable discussions with Gustavo Gioia, Pinaki Chakraborty, Carlo Cesar Zuniga Zamalloa, Patricio Jeraldo, Tuan Tran, John Kolinski, Hamid Kellay, Walter Goldburg, and Grigory Barenblatt. N.G. was partially supported by the University of Illinois. This material is based upon the work supported by the National Science Foundation under Grant No. NSF DMR 06-04435.

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