Structural Stability and Renormalization Group for Propagating Fronts

G. C. Paquette,* Lin-Yuan Chen, Nigel Goldenfeld, and Y. Oono

Department of Physics, Materials Research Laboratory and Beckman Institute, University of Illinois at Urbana-Champaign,
1110 West Green Street, Urbana, Illinois 61801-3080
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A solution to a given equation is structurally stable if it suffers only an infinitesimal change when the equation (not the solution) is perturbed infinitesimally. We have found that structural stability can be used as a velocity selection principle for propagating fronts. We give examples, using numerical and renormalization group methods.

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The steady state equation for a traveling wave propagating into an unstable state does not always uniquely determine the wave speed. Instead there may be multiple stable steady state traveling wave solutions, even though the physical system described by the equation exhibits reproducibly observable behavior corresponding to only one of these solutions [1–6]. In such a situation, it is desirable to formulate a so-called selection principle, which would allow one a priori to distinguish observable from unobservable steady state front solutions without having to solve directly the equation of motion starting from the initial conditions.

For a certain class of equations, rigorous analysis shows how a wide range of physically realizable initial conditions evolve into the selected front, which turns out to be the slowest stable solution allowed by the steady state equation [7]. A physical, heuristic interpretation of this result, known as the linear marginal stability hypothesis, has been proposed and is believed to be applicable in the so-called pulled case, for which the selected speed may be determined by the linear order terms alone [8,9]. However, it is well known that there is another case, the so-called pushed case, where analysis of the linear order terms alone is not sufficient to determine the speed, and the linear marginal stability hypothesis fails [8,9].

The purpose of this Letter is threefold. First, we recall the notion of structural stability—the stability of a front with respect to a perturbation of the governing equation—and argue that only structurally stable fronts are observable. We next show that for structurally stable fronts, a renormalization group (RG) method can be used to compute the change in the front speed when the governing equation is perturbed by a marginal operator. Finally, by combining the structural stability principle with RG, we are able to predict the selected front itself. Our results apply to both the pulled and pushed cases. Roughly speaking, structural stability is an insensitivity to model modifications, whereas the RG may be interpreted as a method to extract the structurally stable behavior of a model [10,11]. Structural modifications of traveling wave equations have been studied previously (e.g., Zeldovich's work on flame propagation [12]) but to our knowledge, structural stability has not previously been proposed as a selection mechanism. RG methods have been used to study the asymptotics of partial differential equations (PDEs) [11,13] and propagating fronts in the Ginzburg-Landau equation [14].

A good model of reproducibly observable physical phenomena must give structurally stable predictions. That is, the observable predictions provided by the model must be stable against "physically small" modifications of the system being modeled. We will quantify below the meaning of the term physically small for a certain class of reaction-diffusion systems. The idea of structural stability used here is close to that proposed by Andronov and Pontrjagin [15] for dynamical systems. In the modeling of natural phenomena, we need not require, as did Andronov and Pontrjagin, the structural stability of the entire model, but only of the solutions corresponding to reproducibly observable phenomena. We call these structurally stable solutions. Our structural stability hypothesis states that only structurally stable solutions of a model represent reproducibly observable phenomena of the system being modeled. This hypothesis is implicit in most mathematical modeling, and indeed often redundant, yet we will demonstrate that, for reaction-diffusion equations, this hypothesis correctly singles out observable propagating fronts. The basic reason for its efficacy in the situations studied here is that the formulation of reaction-diffusion models sometimes inadvertently includes an unphysical feature, although the model is in some sense close to a class of physically correct models.

Consider Fisher's equation [1] on the interval \( -\infty < x < \infty \):

\[
\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + F(\psi),
\]

where \( F \) is a continuous function with \( F(0)=F(1)=0 \). We will usually be interested in boundary conditions where \( \psi \) is zero at one boundary and unity at the other. If \( F \) satisfies the condition \( F(\psi) > 0 \) for all \( \psi \in (0,1) \), then there exists a stable traveling wave solution interpolating between \( \psi = 1 \) and \( \psi = 0 \) with propagation speed \( c \) for each value of \( c \) greater than or equal to some minimum value \( c^* \). The positivity condition on \( F \) stated above together with differentiability of \( F \) at the origin will henceforth be called the AW condition; when it is
satisfied, $c^* \geq \hat{c} = 2\sqrt{F(0)}$. Aronson and Weinberger [7] proved that for (1) with the AW condition satisfied, the selected solution is that with speed $c^*$. In most systems studied by physicists, the minimum wave speed satisfies $c^* = \hat{c}$, which corresponds to the pulled case. Often, the initial conditions decay sufficiently fast (faster than some exponential function) to $\psi = 0$ that the selected wave speed is in fact $c^*$. The pushed case is equivalent to the statement $c^* > \hat{c}$. In this paper, we are concerned not only with Fisher's equation subject to the AW condition, but also with (systems of) reaction-diffusion (semilinear parabolic) equations not satisfying the conditions required for Aronson and Weinberger's rigorous proof, but which still exhibit the selection problem.

It is straightforward to show that all propagating solutions of (1) are structurally stable against $C^1$-small perturbations $\delta F$ of $F$. Unfortunately, reaction-diffusion equations are not in general structurally stable with respect to $C^0$-small perturbations. Consider (1) as describing the propagation of fire along a fuse. $F$ represents the net rate of heat production as a function of temperature $\psi$. The value $\psi = 0$ corresponds to the flash point, and $\psi = 1$ corresponds to the steady burning temperature. It is reasonable that the observable properties of such a front would be insensitive to most small changes to $F$. However, by altering $F$ very near $\psi = 0$ with a $C^0$-small perturbation, $dF/d\psi$ in the neighborhood of $\psi = 0$ can be made arbitrarily large. That is, the rate at which heat production increases as a function of temperature at or near the flash point can be made very large, and this explosive low temperature behavior will travel very rapidly along the fuse.

It is clear then that certain $C^0$-small perturbations are not physically small. This is the case, however, only for perturbations which increase $\sup_{\psi \in (0,1]} |F(\psi)/\psi|$ appreciably for some $\eta > 0$. We will call a $C^0$-small perturbation for which the one-sided bound $\sup_{\psi > 0} |\delta F(\psi)/\psi|$ is less than some small positive number (which tends to zero continuously as the $C^0$ norm of $\delta F$ vanishes) a $p$-small perturbation [16]. The precise form of our structural stability hypothesis for (1) is as follows: Physically realizable solutions are those which are stable with respect to $p$-small structural perturbations. We believe that this hypothesis is correct for other systems, but stress that $p$-small perturbations for other systems may differ from the ones given here.

The ordinary differential equation (ODE) governing the traveling wave front shape $\psi(\xi) = \psi(x,t)$ can be transformed into the equation
\begin{equation}
\dot{p} = -cp - \frac{dU}{dq},
\end{equation}
with the identifications $\xi = x - ct \rightarrow t$, $\psi \rightarrow q$, $d\psi/d\xi \rightarrow q = p$, and $F = dU/dq$. This ODE describes the position $q$ of a unit mass particle subject to a potential $U(q)$ and friction. The coefficient of friction is $c$, the speed of the traveling wave. Traveling wave solutions of (1) interpo-
Let \( \psi_0(x - c_0 t + x_0) \) be a stable traveling front solution of (1) with speed \( c_0 \) and constant of integration \( x_0 \). Let us add a \( p \)-small structural perturbation \( \delta F \) to (1), where its sup-norm \( \| \delta F \| \) is of order \( \epsilon \), a small positive number [21], and assume that in response the front solution is modified to \( \psi_0 + \delta \psi \). Defining \( \xi_0 = x - c_0 t + x_0 \) and linearizing (1) with respect to \( \epsilon \) in the moving frame with velocity \( c_0 \), we formally obtain the following naive perturbation result:

\[
delta \psi(\xi_0, t) = \epsilon e^{-\epsilon \omega(\xi_0)} \int_{t_0}^{t} dt' \int_{-\infty}^{+\infty} d\xi' G(\xi_0, t; \xi', t') e^{\omega(\xi') \delta F(\psi_0(\xi'))} .
\]

(4)

Here \( t_0 \) is a certain time before \( \delta F(\psi_0(\xi_0)) \) becomes nonzero, and \( G \) is the Green's function satisfying

\[
\frac{\partial G}{\partial t} - \mathcal{L} G = \delta(t - t') \delta(\xi - \xi')
\]

(5)

with \( G \to 0 \) in \( |\xi - \xi'| \to \infty \), where

\[
\mathcal{L} = \frac{\partial^2}{\partial \xi^2} + F'(\psi_0(\xi)) - \frac{c_0^2}{4} .
\]

(6)

Formally, \( G \) reads

\[
G(\xi, t; \xi', t') = u_0(\xi) u_0^*(\xi') + \sum \epsilon e^{-\lambda_n(t - t')} u_n(\xi) u_n^*(\xi') ,
\]

(7)

where \( \mathcal{L} u_0 = 0 \), and \( \mathcal{L} u_n = \lambda_n u_n \). The summation symbol, which may imply appropriate integration, is over the spectrum other than the point spectrum \([0]\). Because the system is translationally symmetric, \( u_0 \propto e^{i\phi_0} \psi_0(\xi) \). Because of the known stability of the propagating wave front, the operator \( \mathcal{L} \) is dissipative, so \( 0 \) is the least upper bound of its spectrum. Hence, only \( u_0 \) contributes to the secular term (the term proportional to \( t - t_0 \)) in \( \delta \psi \).

Thus we can write

\[
\psi(x, t) = \psi_0(\xi_0) - \delta c (t - t_0) \psi_0(\xi_0) + (\delta \psi) + O(\epsilon^2)
\]

\[
= \psi_0(\xi) + \epsilon a_1 \psi_0(\xi) - \delta c (t - t_0) \psi_0(\xi) + (\delta \psi) + O(\epsilon^2) .
\]

(10)

where \( \xi = x - c_0 t + x_0(\mu) \). Thus we can choose \( \epsilon a_1 = (\mu - t_0) / \delta c \) to eliminate the divergence to \( O(\epsilon) \). Requiring that \( \psi \) be independent of \( \mu \) gives the RG equation

\[
\frac{\partial \psi}{\partial t} + \delta c \frac{\partial \psi}{\partial \xi} = O(\epsilon^2) .
\]

(12)

Thus the speed of the renormalized wave is indeed \( c_0 + \delta c \). The formula (9) can also be obtained from the solvability condition for the first order correction \( \delta \psi \), and is an example of a very general relation between renormalizability and solvability [23]. Furthermore, (12) corresponds to the amplitude equation describing the slow motion. This relation is also quite general [23].

As an illustration of the use of the renormalized perturbation theory consider the following examples. The first, a pulled case example, is Eq. (1) with the nonlinear operator \( \mathcal{F} = \psi(1 - \psi) \) and the perturbation \( \delta \mathcal{F} = \epsilon \psi(1 - \psi) \). In this trivial case, the exact result is, of course, \( c^* = 2 + \epsilon \). However (9) gives \( c^* = 2 + \frac{\epsilon}{2} \). A more interesting pushed case example is provided by Eq. (3) with \( b \in (0, \frac{1}{2}) \). When \( \gamma = 0 \), we have \( c^*(0) = \sqrt{2b} + 1 / \sqrt{2b} \).

For nonzero \( \gamma \), (9) gives \( c^*(\gamma) = c^*(0) - \gamma c^*(0) \sqrt{2(2b^2 + 1) / 10} \) with \( s = 2c^*(0)/c^*(0) + \sqrt{c^*(0)^2 - 4} \). This agrees well with numerical calculations. For example, this result gives \( c^*(0.08) \approx 2.696 \) for \( b = 0.1 \), while the corresponding value determined numerically [20] is 2.715.

The perturbation theory result (9) can also be used to calculate heuristically the selected speed of the unperturbed system, using the structural stability idea. Within perturbation theory, a necessary and sufficient condition that \( c^* \) be the selected speed is that \( \delta c(c^*) \) be bounded. For example, when \( \mathcal{F} = \psi(1 - \psi) \), the change in the velocity \( \delta c(c) \) is zero as \( \| \delta F \| \to 0 \) for all perturbations \( \delta F \), which are both \( p \) small and differentiable at the origin, only if \( c = c^* = 2 \); for \( c > c^* \) there exist such perturbations for which \( \delta c \) does not vanish. A simple example of the latter is the perturbation \( \delta F = \theta(u - \Delta)(u - \Delta)(1 - u) - u(1 - u) \), as \( \Delta \to 0^+ \).

What is the physical significance of structural stability? Returning to the fuse analogy introduced above, we
can imagine the fuse to be covered with a very thin film of water which quickly evaporates when heated to a temperature slightly above \( \psi \to 0 \); nevertheless, the film suppresses tip ignition. Thus even a small perturbation can destroy (or drastically alter) the tip of a propagating front. For this reason, any front whose behavior is determined by its tip can be destroyed by such a perturbation. If and only if a front’s behavior is independent of the details of its tip can it survive such a perturbation and be structurally stable; hence, a front with \( c > c^* \) is not structurally stable, because no deviation is permitted from the required decay ahead of the front. On the other hand, the front with \( c = c^* \) is insensitve to its tip, as we can see from our explicitly dynamical RG calculation. There, the leading edge is determined by the initial conditions, is not universal, and vanishes as \( \psi \to \infty \). Nevertheless, for sufficiently rapid leading edge decay in space, the asymptotic speed is \( c^* \). Thus \( c^* \) is independent of the details of the leading edge, so that the front is structurally stable \([17,18]\). In future work, we hope to address the question of selection for pattern-forming fronts.

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*Present address: Department of Physics, Kyoto University, Kyoto 606, Japan.

[19] More precisely, for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that for any \( p \)-small \( \epsilon^* \) perturbation, \( |\delta F| < \delta \) implies \( |c^*(F + \delta F) - c^*(F)| < \epsilon \). Here \( \| \| \) is the standard sup-norm, and \( c^*(F) \) stands for the critical frictional coefficient for the force \( F \).
[21] The relationship between \( \epsilon \) and any small parameter in \( \delta F \) may be complicated, as in the examples below Eq. (3); in many cases, \( \delta F \ll \epsilon \).
[22] For Eq. (1) the essential spectrum ranges from \( -\infty \) to \( \max[F'(1), F'(0)] - \epsilon d/\epsilon \). Thus, in the pushed case, 0 is an isolated point.
[24] One may regard our renormalization scheme as separating out the divergence by splitting \( T/\epsilon^2 = (T/L)/L/\epsilon^2 \). The divergence of \( \mu - t_0 \) is then absorbed in \( \psi \).