Structural Stability and Renormalization Group for Propagating Fronts

G. C. Paquette,* Lin-Yuan Chen, Nigel Goldenfeld, and Y. Oono

Department of Physics, Materials Research Laboratory and Beckman Institute, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080

(Received 26 August 1993)

A solution to a given equation is structurally stable if it suffers only an infinitesimal change when the equation (not the solution) is perturbed infinitesimally. We have found that structural stability can be used as a velocity selection principle for propagating fronts. We give examples, using numerical and renormalization group methods.

PACS numbers: 47.20.Ky, 03.40.Kf, 68.10.Gw

The steady state equation for a traveling wave propagating into an unstable state does not always uniquely determine the wave speed. Instead there may be multiple stable steady state traveling wave solutions, even though the physical system described by the equation exhibits reproducibly observable behavior corresponding to only one of these solutions [1-6]. In such a situation, it is desirable to formulate a so-called *selection principle*, which would allow one *a priori* to distinguish observable from unobservable steady state front solutions *without* having to solve directly the equation of motion starting from the initial conditions.

For a certain class of equations, rigorous analysis shows how a wide range of physically realizable initial conditions evolve into the selected front, which turns out to be the slowest stable solution allowed by the steady state equation [7]. A physical, heuristic interpretation of this result, known as the *linear marginal stability hypothesis*, has been proposed and is believed to be applicable in the so-called *pulled case*, for which the selected speed may be determined by the linear order terms alone [8,9]. However, it is well known that there is another case, the so-called *pushed case*, where analysis of the linear order terms alone is not sufficient to determine the speed, and the linear marginal stability hypothesis fails [8,9].

The purpose of this Letter is threefold. First, we recall the notion of *structural stability*—the stability of a front with respect to a perturbation of the governing equation—and argue that only structurally stable fronts are observable. We next show that for structurally stable fronts, a renormalization group (RG) method can be used to compute the change in the front speed when the governing equation is perturbed by a marginal operator. Finally, by combining the structural stability principle with RG, we are able to predict the selected front itself. Our results apply to both the pulled and pushed cases. Roughly speaking, structural stability is an insensitivity to model modifications, whereas the RG may be interpreted as a method to extract the structurally stable behavior of a model [10,11]. Structural modifications of traveling wave equations have been studied previously (e.g., Zel'dovich's work on flame propagation [12]) but to our knowledge, structural stability has not previously been proposed as a selection mechanism. RG methods have been used to study the asymptotics of partial differential equations (PDEs) [11,13] and propagating fronts in the Ginzburg-Landau equation [14].

A good model of reproducibly observable physical phenomena must give *structurally stable* predictions. That is, the observable predictions provided by the model must be stable against "physically small" modifications of the system being modeled. We will quantify below the meaning of the term physically small for a certain class of reaction-diffusion systems. The idea of structural stability used here is close to that proposed by Andronov and Pontrjagin [15] for dynamical systems. In the modeling of natural phenomena, we need not require, as did Andronov and Pontrjagin, the structural stability of the entire model, but only of the *solutions* corresponding to reproducibly observable phenomena. We call these *structurally stable solutions.* Our *structural stability hypothesis* states that only structurally stable solutions of a model represent reproducibly observable phenomena of the system being modeled. This hypothesis is implicit in most mathematical modeling, and indeed often redundant, yet we will demonstrate that, for reaction-diffusion equations, this hypothesis correctly singles out observable propagating fronts. The basic reason for its efficacy in the situations studied here is that the formulation of reaction-diffusion models sometimes inadvertently includes an unphysical feature, although the model is in some sense close to a class of physically correct models.

Consider Fisher's equation [1] on the interval $-\infty < x < \infty$:

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + F(\psi) , \tag{1}$$

where F is a continuous function with $F(0) = F(1) = 0$. We will usually be interested in boundary conditions where ψ is zero at one boundary and unity at the other. If F satisfies the condition $F(\psi) > 0$ for all $\psi \in (0,1)$, then there exists a stable traveling wave solution interpolating between $\psi = 1$ and $\psi = 0$ with propagation speed c for each value of c greater than or equal to some minimum value c^*. The positivity condition on F stated above together with differentiability of F at the origin will henceforth be called the AW condition; when it is

0031-9007/93/72(1)/76(4)$06.00

satisfied, $c^* \geq \hat{c} \equiv 2\sqrt{F'(0)}$. Aronson and Weinberger [7] proved that for (1) with the AW condition satisfied, the selected solution is that with speed c^*. In most systems studied by physicists, the minimum wave speed satisfies $c^* = \hat{c}$, which corresponds to the pulled case. Often, the initial conditions decay sufficiently fast (faster than some exponential function) to $\psi = 0$ that the selected wave speed is in fact c^*. The pushed case is equivalent to the statement $c^* > \hat{c}$. In this paper, we are concerned not only with Fisher's equation subject to the AW condition, but also with (systems of) reaction-diffusion (semilinear parabolic) equations not satisfying the conditions required for Aronson and Weinberger's rigorous proof, but which still exhibit the selection problem.

It is straightforward to show that all propagating solutions of (1) are structurally stable against C^1-small perturbations δF of F. Unfortunately, reaction-diffusion equations are not in general structurally stable with respect to C^0-small perturbations. Consider (1) as describing the propagation of fire along a fuse. F represents the net rate of heat production as a function of temperature ψ. The value $\psi = 0$ corresponds to the flash point, and $\psi = 1$ corresponds to the steady burning temperature. It is reasonable that the observable properties of such a front would be insensitive to most small changes to F. However, by altering F very near $\psi = 0$ with a C^0-small perturbation, $dF/d\psi$ in the neighborhood of $\psi = 0$ can be made arbitrarily large. That is, the rate at which heat production increases as a function of temperature at or near the flash point can be made very large, and this explosive low temperature behavior will travel very rapidly along the fuse.

It is clear then that certain C^0-small perturbations are not physically small. This is the case, however, only for perturbations which increase $\sup_{\psi \in (0,\eta]}[F(\psi)/\psi]$ appreciably for some $\eta > 0$. We will call a C^0-small perturbation for which the one-sided bound $\sup_{\psi > 0}[\delta F(\psi)/\psi]$ is less than some small positive number (which goes to zero continuously as the C^0 norm of δF vanishes) a *p-small* perturbation [16]. The precise form of our structural stability hypothesis for (1) is as follows: Physically realizable solutions are those which are stable with respect to p-small structural perturbations. We believe that this hypothesis is correct for other systems, but stress that p-small perturbations for other systems may differ from the ones given here.

The ordinary differential equation (ODE) governing the traveling wave front shape $\psi(\xi) = \psi(x,t)$ can be transformed into the equation

$$\dot{p} = -cp - \frac{dU}{dq}, \tag{2}$$

with the identifications $\xi \equiv x - ct \to t$, $\psi \to q$, $d\psi/d\xi \to \dot{q} \equiv p$, and $F \equiv dU/dq$. This ODE describes the position q of a unit mass particle subject to a potential $U(q)$ and friction. The coefficient of friction is c, the speed of the traveling wave. Traveling wave solutions of (1) interpolating between the fixed points $\psi = 0$ and $\psi = 1$ correspond in this particle analogy to trajectories which begin at the maximum of U located at $q = 1$, with zero kinetic energy, and which terminate at the origin. Those trajectories which correspond to stable solutions of the original PDE are those for which q never changes sign.

If the origin is not an isolated local minimum of U, there is only one value of the coefficient of friction which allows the particle to stop here without overshooting. If the origin is an isolated local minimum of the potential, there is a critical value c^* of the frictional coefficient. For all smaller values, the particle overshoots the origin at least once, while for a continuous set of larger values, it converges to the origin as $t \to \infty$ without overshooting. For both types of systems, we define c^* to be the smallest value (unique value in the former case) for which the particle approaches the origin in the $t \to \infty$ limit without overshooting.

We have proven that c^* is continuously dependent on the continuous function F [17-19]. Thus because we can make the origin a local maximum of U with arbitrarily small C^0 perturbations (in fact, p-small perturbations), and because c^* is the only value of c for which the particle stops at the origin (without overshooting) in this case, the continuity of c^* implies that only this critical value is structurally stable. Therefore, our structural stability hypothesis asserts that c^* is the unique observable front propagation speed of the original PDE.

For AW-type equations, it has been proven that c^* is indeed the selected speed. Our structural stability hypothesis is thus rigorously correct in this case. For non-AW-type equations, however, there is no proof that this minimum value is selected. We have therefore performed numerical studies to test our conclusions for non-AW equations as well as for systems of coupled reaction-diffusion equations [17,18]. An example is the equation studied by van Saarloos [20]:

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} - \gamma \frac{\partial^4 \psi}{\partial x^4} + \frac{\psi}{b}(b+\psi)(1-\psi), \tag{3}$$

where $\gamma < \frac{1}{12}$. For $0 < b < \frac{1}{2}$, the front is pushed, whereas for $\frac{1}{2} < b < 1$ the front is pulled. We replaced the potential term by $\theta(\psi-\Delta)(\psi-\Delta)(1-\psi)(\psi+b)/b$, where θ is the step function, and solved the steady state equation numerically for a sequence of potentials where $\Delta \to 0^+$. For both pulled and pushed cases, the unique selected velocity converged to that dynamically selected.

We now proceed to show how RG can be used to compute the change in the front speed when an equation, whose structurally stable exact solution is known, is perturbed by a p-small operator. Introducing new variables $X \equiv e^x$ and $T \equiv e^t$, the propagating front solution becomes a similarity solution $\psi(x-ct) = \Phi(XT^{-c})$ and c may be interpreted as an anomalous dimension [11,13,14]. The preceding arguments imply that for structurally stable fronts p-small C^0 perturbations are marginal, but general C^0 perturbations are relevant.

Let $\psi_0(x-c_0t+x_0)$ be a stable traveling front solution of (1) with speed c_0 and constant of integration x_0. Let us add a p-small structural perturbation δF to (1), where its sup-norm $\|\delta F\|$ is of order ϵ, a small positive number [21], and assume that in response the front solution is modified to $\psi_0+\delta\psi$. Defining $\xi_0 \equiv x-c_0t+x_0$ and linearizing (1) with respect to ϵ in the moving frame with velocity c_0, we formally obtain the following naive perturbation result:

$$\delta\psi(\xi_0,t)=e^{-c_0\xi_0/2}\int_{t_0}^{t}dt'\int_{-\infty}^{+\infty}d\xi' G(\xi_0,t;\xi',t')e^{c_0\xi'/2}\delta F(\psi_0(\xi'))\,. \tag{4}$$

Here t_0 is a certain time before $\delta F(\psi_0(\xi_0))$ becomes nonzero, and G is the Green's function satisfying

$$\frac{\partial G}{\partial t}-\mathcal{L}G=\delta(t-t')\delta(\xi-\xi') \tag{5}$$

with $G\to 0$ in $|\xi-\xi'|\to\infty$, where

$$\mathcal{L}\equiv\frac{\partial^2}{\partial\xi^2}+F'(\psi_0(\xi))-\frac{c_0^2}{4}\,. \tag{6}$$

Formally, G reads

$$G(\xi,t;\xi',t')=u_0(\xi)u_0^*(\xi')+\sum e^{-\lambda_n(t-t')}u_n(\xi)u_n^*(\xi')\,, \tag{7}$$

where $\mathcal{L}u_0=0$, and $\mathcal{L}u_n=\lambda_n u_n$. The summation symbol, which may imply appropriate integration, is over the spectrum other than the point spectrum $\{0\}$. Because the system is translationally symmetric, $u_0\propto e^{c_0\xi/2}\psi_0'(\xi)$. Because of the known stability of the propagating wave front, the operator $\mathcal{L}$ is dissipative, so 0 is the least upper bound of its spectrum. Hence, only u_0 contributes to the secular term (the term proportional to $t-t_0$) in $\delta\psi$. Thus we can write

$$\delta\psi=-(t-t_0)\delta c\psi_0'(\xi)+(\delta\psi)_r\,, \tag{8}$$

where $(\delta\psi)_r$ is the bounded piece (regular part), and

$$\delta c=-\lim_{l\to\infty}\frac{\int_{-l}^{+l}d\xi e^{c_0\xi}\psi_0'(\xi)\delta F(\psi_0(\xi))}{\int_{-l}^{+l}d\xi e^{c_0\xi}\psi_0'^2(\xi)}\,. \tag{9}$$

This formula is easily justified if 0 is isolated from the essential spectrum of $\mathcal{L}$. In the pulled case this condition is not satisfied [5,22], but a more detailed argument shows that (9) remains valid [23].

One might immediately guess that δc is the $O(\epsilon)$ change in the front speed, but the naive perturbation theory is not controlled, due to the secular divergence as $t_0\to-\infty$. This divergence can be controlled by perturbatively renormalizing the constant of integration: $x_0=x_0^R+Z$, where x_0^R is the finite observable counterpart of x_0, the renormalization constant $Z=\sum_1^\infty a_n(t_0,\mu)\epsilon^n$, and the coefficients a_n are chosen order by order in ϵ to eliminate the secular divergence. The quantity μ parametrizes the family of solutions to the unperturbed equation; it corresponds to the arbitrary renormalization point in the Gell-Mann–Low RG [24].

To order ϵ the solution ψ is given by

$$\psi(x,t)=\psi_0(\xi_0)-\delta c(t-t_0)\psi_0'(\xi_0)+(\delta\psi)_r+O(\epsilon^2) \tag{10}$$

$$=\psi_0(\xi)+\epsilon a_1\psi_0'(\xi)-\delta c(t-t_0)\psi_0'(\xi)+(\delta\psi)_r+O(\epsilon^2)\,, \tag{11}$$

where $\xi\equiv x-c_0t+x_0^R(\mu)$. Thus we can choose $\epsilon a_1=(\mu-t_0)\delta c$ to eliminate the divergence to $O(\epsilon)$. Requiring that ψ be independent of μ gives the RG equation

$$\frac{\partial\psi}{\partial t}+\delta c\frac{\partial\psi}{\partial\xi}=O(\epsilon^2)\,. \tag{12}$$

Thus the speed of the renormalized wave is indeed $c_0+\delta c$. The formula (9) can also be obtained from the solvability condition for the first order correction $\delta\psi$, and is an example of a very general relation between renormalizability and solvability [23]. Furthermore, (12) corresponds to the amplitude equation describing the slow motion. This relation is also quite general [23].

As an illustration of the use of the renormalized perturbation theory consider the following examples. The first, a pulled case example, is Eq. (1) with the nonlinear operator $F=\psi(1-\psi)$ and the perturbation $\delta F=\epsilon\psi(1-\psi)$. In this trivial case, the exact result is, of course, $c^*=2\sqrt{1+\epsilon}$, whereas (9) gives $c^*\simeq 2+\epsilon$. A more interesting pushed case example is provided by Eq. (3) with $b\in(0,\frac{1}{2})$. When $\gamma=0$, we have $c^*(0)=\sqrt{2b}+1/\sqrt{2b}$. For nonzero γ, (9) gives $c^*(\gamma)=c^*(0)-\gamma c^*(0)^3s^4(2s^2+1)/10$ with $s\equiv 2c^*(0)/[c^*(0)+\sqrt{c^*(0)^2-4}]$. This agrees well with numerical calculations. For example, this result gives $c^*(0.08)\simeq 2.696$ for $b=0.1$, while the corresponding value determined numerically [20] is 2.715.

The perturbation theory result (9) can also be used to calculate heuristically the selected speed of the unperturbed system, using the structural stability idea. Within perturbation theory, a necessary and sufficient condition that c^* be the selected speed is that $\delta c(c^*)$ be bounded. For example, when $F=\psi(1-\psi)$, the change in the velocity $\delta c(c)$ is zero as $\|\delta F\|\to 0$ for all perturbations δF, which are both p small and differentiable at the origin, only if $c=c^*=2$; for $c>c^*$ there exist such perturbations for which δc does not vanish. A simple example of the latter is the perturbation $\delta F=\theta(u-\Delta)(u-\Delta)(1-u)-u(1-u)$, as $\Delta\to 0^+$.

What is the physical significance of structural stability? Returning to the fuse analogy introduced above, we can imagine the fuse to be covered with a very thin film of water which quickly evaporates when heated to a temperature slightly above $\psi=0$; nevertheless, the film suppresses tip ignition. Thus even a small perturbation can destroy (or drastically alter) the tip of a propagating front. For this reason, any front whose behavior is determined by its tip can be destroyed by such a perturbation. If and only if a front's behavior is independent of the details of its tip can it survive such a perturbation and be structurally stable; hence, a front with $c > c^*$ is not structurally stable, because no deviation is permitted from the required decay ahead of the front. On the other hand, the front with $c=c^*$ is insensitive to its tip, as we can see from our explicitly dynamical RG calculation. There, the leading edge is determined by the initial conditions, is not universal, and vanishes as $t \to \infty$. Nevertheless, for sufficiently rapid leading edge decay in space, the asymptotic speed is c^*: Thus c^* is independent of the details of the leading edge, so that the front is structurally stable [17,18]. In future work, we hope to address the question of selection for pattern-forming fronts.

The authors are grateful to Paul Newton for valuable discussions. This work is, in part, supported by the National Science Foundation Grant No. NSF-DMR-90-15791. L.Y.C. acknowledges the support of Grant No. NSF-DMR-89-20538, administered through the University of Illinois Materials Research Laboratory. G.C.P. acknowledges support from the Japanese Society for the Promotion of Science.

*Present address: Department of Physics, Kyoto University, Kyoto 606, Japan.

[1] R. A. Fisher, Ann. Eugenics **7**, 355 (1937).

[2] A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov, Bull. Univ. Moskou. Ser. Internat. Sec. A **1**, 1 (1937).

[3] R. FitzHugh, J. Biophys. **1**, 445 (1961); J. Nagumo, S. Arimoto, and S. Yoshizawa, Proc. IRE **50**, 2061 (1962).

[4] H. Meinhardt, *Models for Biological Pattern Formation* (Academic, London, 1982).

[5] P. Collet and J.-P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton Univ. Press, Princeton, NJ, 1990).

[6] For an early review of the dendritic growth problem see J. S. Langer, Rev. Mod. Phys. **52**, 1 (1980); recent work is reviewed by J. S. Langer, in *Chance and Matter*, edited by J. Souletie, J. Vannimenus, and R. Stora (North-Holland, Amsterdam, 1987).

[7] D. G. Aronson and H. F. Weinberger, in *Partial Differential Equations and Related Topics*, edited by J. A. Goldstein (Springer, Heidelberg, 1975); H. F. Weinberger, SIAM J. Math. Anal. **13**, 3 (1982); a pedagogical introduction to this literature is given by P. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics Vol. 28, edited by S. Levin (Springer, New York, 1979).

[8] J. S. Langer and H. Müller-Krumbhaar, Phys. Rev. A **27**, 499 (1983); E. Ben-Jacob, H. R. Brand, G. Dee, L. Kramer, and J. S. Langer, Physica (Amsterdam) **14D**, 348 (1985); W. van Saarloos, Phys. Rev. Lett. **58**, 24 (1987); Phys. Rev. A **37**, 1 (1988).

[9] G. Dee and J. S. Langer, Phys. Rev. Lett. **50**, 6 (1983); G. Dee, J. Stat. Phys. **39**, 705 (1985); Physica (Amsterdam) **15D**, 295 (1985); G. Dee and W. van Saarloos, Phys. Rev. Lett. **60**, 2641 (1988).

[10] Y. Oono, Adv. Chem. Phys. **61**, 301 (1985); Kobunshi **28**, 781 (1979).

[11] N. D. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992), Chap. 10.

[12] Ya. B. Zel'dovich, Zhurn. Fiz. Khimii. **22**, 27 (1948); see also G. I. Barenblatt, *Similarity, Self-Similarity, and Intermediate Asymptotics* (Consultants Bureau, New York, 1979), p. 111.

[13] N. Goldenfeld, O. Martin, and Y. Oono, J. Sci. Comput. **4**, 355 (1989); N. Goldenfeld, O. Martin, Y. Oono, and F. Liu, Phys. Rev. Lett. **64**, 1361 (1990); N. Goldenfeld and Y. Oono, Physica (Amsterdam) **177A**, 213 (1991); N. Goldenfeld, O. Martin, and Y. Oono, in *Proceedings of the NATO Advanced Research Workshop on Asymptotics Beyond All Orders,* edited by S. Tanveer (Plenum, New York, 1992); L. Y. Chen, N. D. Goldenfeld, and Y. Oono, Phys. Rev. A **44**, 6544 (1991); I. S. Ginzburg, V. M. Entov, and E. V. Theodorovich, J. Appl. Math. Mech. **56**, 59 (1992); L. Y. Chen and N. D. Goldenfeld, Phys. Rev. A **45**, 5572 (1992); J. Bricmont, A. Kupiainen, and G. Lin, Commun. Pure Appl. Math. (to be published).

[14] J. Bricmont and A. Kupiainen, Commun. Math. Phys. **150**, 193 (1992).

[15] A. Andronov and L. Pontrjagin, Dokl. Akad. Nauk. SSSR **14**, 247 (1937).

[16] This condition is sufficient to ensure that the quantity

$$\sup_{\eta>0}\left\{\sup_{\psi\in(0,\eta]}\left[\frac{F(\psi)+\delta F(\psi)}{\psi}\right]-\sup_{\psi\in(0,\eta]}\left[\frac{F(\psi)}{\psi}\right]\right\}$$

is less than the same small number.

[17] G. C. Paquette and Y. Oono (to be published).

[18] G. C. Paquette, thesis, University of Illinois at Urbana-Champaign, Department of Physics, 1992.

[19] More precisely, for any $\epsilon>0$ there is $\delta>0$ such that for any p-small C^0 perturbation, $\|\delta F\|<\delta$ implies $|c^*(F+\delta F)-c^*(F)|<\epsilon$. Here $\|\ \|$ is the standard sup-norm, and $c^*(F)$ stands for the critical frictional coefficient for the force F.

[20] W. van Saarloos, Phys. Rev. A **39**, 6367 (1989).

[21] The relationship between ϵ and any small parameter in δF may be complicated, as in the example below Eq. (3); in many cases, $\delta F \propto \epsilon$.

[22] For Eq. (1) the essential spectrum ranges from $-\infty$ to $\max[F'(1),F'(0)]-c_0^2/4$. Thus, in the pushed case, 0 is an isolated point.

[23] L. Y. Chen, N. D. Goldenfeld, and Y. Oono (to be published); L. Y. Chen, N. D. Goldenfeld, G. Paquette, and Y. Oona, Physica A (to be published).

[24] One may regard our renormalization scheme as separating out the divergence by splitting $t-t_0=t-\mu-(t_0-\mu)$, which corresponds, in the equivalent similarity solution problem, to writing $T/e^{t_0}=(T/L)L/e^{t_0}$. The divergence of $\mu-t_0$ is then absorbed in ψ.