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Renormalisation group theory for two problems in linear continuum mechanics

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We consider the problems of: (1) the stress field in an infinite wedge when a moment is applied at the tip, and (2) the flow of an inviscid incompressible fluid past an infinite wedge. Both problems are linear, but nevertheless exhibit anomalous dimensions for wedge angles, α, larger than a critical value α_c. In (1) it is the stress field and in (2) it is the velocity potential which can have anomalous power law behaviour in distance from the tip. We use these problems as simple examples to illustrate how partial differential equations can be solved using the renormalisation group.

1. Introduction

In a series of recent papers [1–4], we have begun to develop a renormalisation group (RG) theory for the asymptotics of partial differential equations (PDEs). In particular, we showed, by explicit example, that certain problems encountered in porous medium flow may exhibit anomalous dimensions in the long time limit, and, furthermore, that the appropriate exponents could be computed using both perturbative RG techniques and Wilson's RG. Our work establishes a direct correspondence between intermediate asymptotics (IA) [5] and RG.

The purpose of the present paper is to elaborate further on this correspondence, by solving two *linear* problems in continuum mechanics, discussed by Barenblatt [5]. These problems are simple enough that they permit an exact solution, but nevertheless are rich enough to exhibit interesting asymptotics. The application of the RG to these problems does nevertheless add insight into the asymptotics, and serves to illustrate what we believe to be the physical meaning of the RG. This work is a response to a very stimulating challenge by Professor Barenblatt [6].

It is both appropriate, and with great pleasure, that we dedicate this article to Michael Fisher on this auspicious occasion. Michael has always been clear about both the physics of RG and the connection with asymptotics. We have learned much from him. We send him our very best wishes for his birthday and for the future.

2. Observables, renormalisation and asymptotics

In this paper, we consider two linear problems in continuum mechanics, suggested to us by Barenblatt [6]. Since these linear problems can be solved exactly, it might be thought that the RG does not yield any new quantitative information, in these cases. However, the RG method does make a conceptually important contribution, by justifying the functional form assumed in the IA approach. This is important, even for the simple problems treated in this paper, because the RG, together with a perturbative analysis (which is often not hard to obtain), allows one to compute the anomalous dimension in a series form just as in the case of the ϵ-expansion [7] in critical phenomena. We emphasise, however, that the RG is essentially non-perturbative, and that renormalisation, as discussed here, occurs even outside perturbation theory [3,4].

The two problems which we discuss [5] concern the stress field of an infinite wedge at whose tip is applied a moment (the Sternberg–Koiter problem [8]), and the flow of an inviscid incompressible fluid past an infinite wedge. Both of these cases are linear problems, but naïve dimensional analysis does not work, and strange power laws emerge. In this section, we analyze both of the problems from the RG point of view, paying attention to the question of which quantities are actually observable. These considerations enable us to anticipate the number and nature of the renormalisation constants required to renormalise the perturbation theory. The requirement that the physics on large length scales is well defined as the regularisation parameter is taken to be arbitrarily small, dictates the functional form of the quantity we compute.

2.1. The Sternberg–Koiter problem

The Sternberg–Koiter problem in its original form (as is explained in ref. [5], ch. 10) is to compute the stress tensor of an infinite two-dimensional wedge, whose dihedral angle is 2α, under the application of a couple of moment M at the tip of the wedge. We use the polar coordinate system (r, θ). We wish to study the stress σ_{rr}, which is directly observable. There are five quantities in the problem: M, α, r, θ and the stress σ_{rr}. Dimensional analysis (DA) implies the following functional relation [5]:

$$\sigma_{rr} = \frac{M}{r^2}\phi_0(\theta, \alpha)\,, \tag{2.1}$$

where ϕ_0 is a function, whose explicit form has been given by Carothers [9] and Inglis [10]:

$$\phi_0 = \frac{2M\sin(2\theta)}{\sin(2\alpha) - 2\alpha\cos(2\alpha)}\,. \tag{2.2}$$

Unfortunately, the denominator of (2.2) vanishes when the dihedral angle

$\alpha=\alpha_c\approx 0.71514839\pi$, leading to a divergent stress field at a dihedral angle which is perfectly acceptable physically. Sternberg and Koiter analyzed this "paradox" by regularising the problem: they introduced a more detailed microscopic model of the physical problem, taking into account the distribution of forces near the tip of the wedge. In other words, they attributed the failure of the macroscopic elasticity theory calculation to the omission of certain microscopic parameters.

This procedure led them to a satisfactory solution, which is interpreted by Barenblatt from the IA point of view. Barenblatt explains that there is a small cutoff length scale, r_0, near the tip, over which act the forces generating the moment M. This introduces a new length scale, which must be taken into account when performing dimensional analysis. This revises (2.1), which now becomes

$$\sigma_{rr}=\frac{M}{r^2}\phi\left(\frac{r}{r_0},\theta,\alpha\right). \tag{2.3}$$

Next, Barenblatt assumes that in the $r/r_0\to\infty$ limit, the scaling function ϕ is singular for $\alpha>\alpha_c$ in the following way:

$$\sigma_{rr}\sim\frac{M}{r^2}\left(\frac{r_0}{r}\right)^{\lambda}\phi(\theta,\alpha)\qquad\text{as}\quad\frac{r}{r_0}\to\infty\,. \tag{2.4}$$

Here, λ is an exponent to be determined. In general, such exponents are determined by substituting the ansatz (2.4) into the original PDE. Only for special (sometimes unique) values of λ can all the boundary conditions be satisfied, and this solvability condition determines λ.

Let us now consider this problem from the RG point of view. It is possible to proceed blindly with the problem, and calculate (e.g. in perturbation theory), introducing renormalisation constants where necessary, in order to get a finite theory. An example of this is the calculation presented in ref. [2], where the interpretation of the renormalisation procedure is provided post hoc. On the other hand, it is useful to anticipate the behaviour of the theory, using physical arguments, as explained in ref. [1], and this is what we will now do with the problem at hand.

The crucial question to ask is: which quantities are observable (under given conditions) and which are not. Among the five quantities considered in the *original* formulation of the problem, r, θ, and α are obviously directly observable. The stress tensor has some ambiguity due to gauge invariance [11], but it is also essentially observable. The moment M is concentrated "at the tip of the wedge" in the original formulation of the problem, and over a distance r_0 in the regularised version of the problem. The original formulation of the problem is a degenerate limit of the regularised problem, but supposedly corresponds to the description of what we are able to observe in the laboratory. Thus, if M_0 is the microscopic moment *applied near the tip* in the regularised problem, the moment M *applied at the tip* in the original formulation is a renormalised counterpart of M_0. The value of M_0 will in principle depend upon the regularisation r_0. This means that as the degenerate limit $r_0 \to 0$ is taken, there is nothing that guarantees that M and M_0 will be the same; in general, they will not be. The concept of a moment applied precisely at the tip is a fiction due to the macroscopic (or phenomenological) description that we have chosen.

To summarise, M_0 and r_0 are the bare parameters of the problem, which are not accessible to our chosen level of description. The usual hypothesis of phenomenology is that there is a closed functional relation among directly observable and phenomenological quantities, and this should be true irrespective of the microscopic details. In particular, this should be true in the limit $r_0 \to 0$. The observable M is related to the bare M_0 by the relation $M = ZM_0$. The renormalisation constant Z is dimensionless, but must depend upon r_0, because M_0 depends upon r_0 and M does not. Dimensional analysis then requires that there be *another* length in the problem, which we will call L, introduced in order to ensure that Z is dimensionless. This is the length that sets the scale, in the language of statistical field theory, and is arbitrary. Thus

$$\sigma_{rr} = \frac{Z(r_0/L)M_0}{r^2}\,\bar{\phi}\left(\frac{r}{r_0}, \theta, \alpha\right), \tag{2.5}$$

with $\bar{\phi}$ being some function to be determined. We can write this as

$$\sigma_{rr} = \frac{Z(r_0/L)M_0}{r^2}\,\phi\left(\frac{r}{L}, \theta, \alpha\right), \tag{2.6}$$

where ϕ is another function to be determined.

We may use the Gell-Mann–Low argument [12] to establish the functional form of σ_{rr}. Since L is not present in the original specification of the problem, it should not be present in the solution of the problem. Hence, we arrive at the RG equation:

$$L\frac{\partial \sigma_{rr}}{\partial L} = 0, \tag{2.7}$$

and thus

$$-\lambda\phi + \rho\frac{\partial \phi}{\partial \rho} = 0, \tag{2.8}$$

where $\rho \equiv r/L$, and

$$\lambda \equiv -\frac{\partial \ln Z}{\partial \ln L}. \tag{2.9}$$

All the partial derivatives above are taken with bare parameters kept constant. Solving this linear partial differential equation, we get

$$\sigma_{rr} = \frac{M}{r^2}\left(\frac{L}{r}\right)^{\lambda}\phi(\theta, \alpha). \tag{2.10}$$

This is the form (2.4) postulated by the IA approach.

2.2. *Ideal flow past a wedge*

This is the problem discussed in the last section of ch. 10 of Barenblatt's book [5]. Consider an infinite wedge with dihedral angle 2α whose symmetry axis is along the x-axis, with the tip at the origin. The problem is to find the velocity field of an ideal incompressible fluid past this wedge with the condition that as $x\to-\infty$, the velocity is U parallel to the x-axis. Since this is a two-dimensional problem, we need only calculate the velocity potential φ. We use the polar coordinate system (r, θ). Thus there are five quantities in the original specification of the problem: r, θ, α, U and φ. Dimensional analysis tells us that [5]

$$\varphi = Ur\Phi(\theta, \alpha)\,, \tag{2.11}$$

where Φ is a function to be determined.

In fact, for $\alpha > 0$, *there is no solution of this form*! This follows from the fact that φ satisfies Laplace's equation. The ansatz (2.11) implies that $\Phi = A\cos\theta + B\sin\theta$, where A and B are determined by the boundary conditions $\partial_n\varphi = 0$ for $\theta = \pm\alpha$. These boundary conditions do not lead to a non-trivial solution for Φ.

Barenblatt regularises the problem by considering instead a wedge of finite length L_0. For the finite length problem, dimensional analysis dictates that

$$\varphi = Ur\Phi(r/L_0, \theta, \alpha)\,. \tag{2.12}$$

Then he assumes the following IA form in the $r/L_0 \to 0$ limit:

$$\varphi = Ur\left(\frac{r}{L_0}\right)^{\lambda}\Phi(\theta, \alpha)\,. \tag{2.13}$$

Let us now consider this problem from the RG point of view. Again, it is important to know what is directly observable at the given level of description. Unobservable quantities are always at the very microscopic scale as in the Sternberg–Koiter problem and critical phenomena or at the very large scale. In the present flow problem, it is tempting to relate unobservable quantities to the tip of the wedge as in the former problem. But in this problem, there is no obvious candidate parameter depending upon the tip. The parameters r, θ and α are obviously unambiguous observables. The velocity potential itself is not directly observable, but it is straightforwardly related to the velocity field, which is directly observable. Hence, the only remaining candidate for the unobservable quantity is U. For a wedge of finite extent, the velocity "at infinity" has a direct meaning. But in the limit of an infinite wedge, this is not the case. Thus for the regularised problem, with a wedge of extent L_0, we denote the speed at infinity by U_0. Thus U_0 and L_0 are the bare quantities in the present problem. Renormalizing this problem as $L_0 \to \infty$ introduces an arbitrary length scale, as before, and it is straightforward to obtain the final result

$$\varphi = Ur\left(\frac{r}{L}\right)^{\lambda} \Phi(\theta, \alpha) . \tag{2.14}$$

This is the functional form used in Barenblatt's book.

3. Conclusion

In this paper, we have seen that simple physical insight, together with RG arguments, suffice to predict the form of the asymptotics, even when naïve dimensional analysis fails. Just as in field theory and statistical mechanics, this prediction of the RG has two important implications. The first is that perturbation theory about the solution expected on the ground of naïve dimensional analysis must be divergent. The second implication is that this divergent perturbation series, when combined with the RG, yields an expansion for the anomalous dimensions. For the problems under consideration here, we will simply quote the final result; the details contain many subtleties and will be published separately.

For the Sternberg–Koiter problem, we find that the anomalous dimension of (2.10) is given by

$$\lambda = -2\epsilon + \mathcal{O}(\epsilon^2) , \tag{3.1}$$

where $\epsilon = (\alpha - \alpha_c)/\alpha_c$, which agrees with the result given in ref. [5] to order ϵ. In practice, this linear approximation is quantitatively reliable only when δ/α_c is less than of order 10^{-3}. For the problem of fluid flow around a wedge, we find that

$$\lambda = \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2) , \tag{3.2}$$

which is indeed the correct result to first order in α.

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