ASYMPTOTICS OF PARTIAL DIFFERENTIAL EQUATIONS AND THE RENORMALISATION GROUP

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I. INTRODUCTION

It is well-known that the asymptotics of partial differential equations (PDEs) may often be found from consideration of similarity solutions. In the examples usually encountered, the combinations of variables making up the similarity variables may be deduced using dimensional analysis; typically, the similarity variables are products of variables raised to rational fraction powers. It is not so widely appreciated, however, that there is a large class of problems where the similarity variables cannot be deduced from dimensional analysis. As Barenblatt has emphasized, such problems are neither rare nor pathological, but occur in many situations of physical interest, for example, in continuum mechanics.[1]

The purpose of this article is to show how to obtain the asymptotics of PDEs, even in cases where dimensional analysis fails, using the renormalisation group (RG).[2] Renormalisation and the RG were originally developed to treat the divergences arising in the perturbation series of quantum electrodynamics, and, following the work of Kadanoff, Wilson and others, have found extensive application in later quantum field theories[3] and statistical mechanics.[4] Although it has been known for some time that field theories--be they quantum or statistical--are equivalent to stochastic partial differential equations[5], it is only recently that it was shown how to use RG techniques for partial differential equations without noise.[2] This article not only summarizes this development, but also emphasizes the connection with the problems of velocity selection in dendritic growth and asymptotics beyond

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all orders, discussed elsewhere in this volume. This connection arises because a travelling wave solution of a PDE in one space dimension, x, and time, t, of the form

$$u(x,t) = f(x - vt) \tag{1}$$

may be mapped by the substitutions $x = \log X$, $t = \log T$ into a similarity solution of the form

$$u(X,T) = T^{\alpha} g(XT^{\beta}) \tag{2}$$

with $\alpha = 0$, $\beta = -v$. The goal of this work, then, is to calculate exponents such as α, β, v and the associated scaling functions f, g. Examples of physical interest occur in (e.g.) elasticity theory, shock wave dynamics, flame propagation, and flow in porous media.[1]

The application of the RG to partial differential equations has implications for the way in which systems approach thermodynamic equilibrium, and is almost certainly relevant to theoretical attempts to account for the prevalence of dynamical scaling. This connection has been discussed in a recent article which complements this one.[6]

2. ASYMPTOTICS OF THE DIFFUSION EQUATION

We begin with an elementary example, which shows explicitly how similarity solutions are relevant for asymptotics. Consider the initial value diffusion problem

$$\partial_t u = \frac{1}{2}\partial_x^2 u \, , \qquad -\infty < x < \infty \tag{3}$$

with $u \to 0$ as $|x| \to \infty$ and initial condition

$$u(x,0) = \frac{A_0}{(2\pi \ell^2)^{1/2}} \, e^{-x^2/2\ell^2} \, . \tag{4}$$

The solution after time t is

$$u(x,t) = \frac{A_0}{2\pi(t + \ell^2)^{1/2}} \, e^{-x^2/2(t + \ell^2)} \, . \tag{5}$$

Note that the dynamics conserves $\int u(x,t)\, dx$. For long times

$$u(x,t) \to \frac{A_0}{(2\pi t)^{1/2}} \, e^{-x^2/2t} \qquad \text{as } t \to \infty. \tag{6}$$

This long time behaviour can also be obtained by keeping t fixed, but letting the width of the initial distribution vanish:

$$u(x,t) \rightarrow \frac{A_0}{(2\pi t)^{1/2}}\, e^{-x^2/2t} \qquad \text{as } \ell \rightarrow 0. \tag{7}$$

We conclude that the long-time behaviour of the initial value problem is given by the degenerate limit of the initial value problem when $\ell \rightarrow 0$, i.e. the similarity solution corresponding to a delta-function initial condition.

In this example, the mathematical results follow "common sense" intuition: at very long times, the distribution should be insensitive to the initial condition. That is, when

$$\langle x^2 \rangle \equiv \int x^2 u(x,t)\, dx >> \ell^2 \tag{8}$$

we expect that the solution u(x,t) should not depend on ℓ, and thus we can safely take the limit $\ell \rightarrow 0$ whilst maintaining the conservation of $\int u(x,t)dx$.

This sort of "common sense" argument is encountered frequently. Suppose a physical problem has been cast in dimensionless form, and the relationship between the dimensionless parameters $\Pi, \Pi_0, \Pi_1, \ldots \Pi_n$ is written as

$$\Pi = f(\Pi_0, \Pi_1, \ldots, \Pi_n). \tag{9}$$

Then "common sense" intuition states that as (e.g.) $\Pi_0 \rightarrow 0$, $\Pi \rightarrow f(0, \Pi_1, \ldots \Pi_n)$. In the diffusion equation example,

$$\Pi = \frac{u}{Q}\sqrt{t}\,; \qquad \Pi_0 = \frac{\ell}{\sqrt{t}}\,; \qquad \Pi_1 = \frac{x}{\sqrt{t}}\,. \tag{10}$$

3. ASYMPTOTICS OF THE SECOND KIND

Barenblatt[1] has pointed out that there are a wide class of problems where "common sense" intuition fails, because the limit $\Pi_0 \rightarrow 0$ is singular. He uses the term "intermediate asymptotics of the first kind" to denote the case when the limit $\Pi_0 \rightarrow 0$ is regular, and the term "intermediate asymptotics of the second kind" to denote the case

$$\Pi \sim \Pi_0^{-\alpha}\, g\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}} \cdots \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right) \tag{11}$$

as $\Pi_0 \rightarrow 0$. All other possibilities are included under the category "third kind." The use of the term "intermediate asymptotics" has the connotation "prior to the final state of the system", which in the diffusion equation example is $u(x, \infty) = 0$.

When the asymptotics is of the second kind, a number of exponents, α, $\alpha_1, \ldots \alpha_n$ are introduced. These cannot be determined by dimensional analysis, since the Π's are already dimensionless. Usually, the exponents are found to satisfy a non-linear eigenvalue equation, obtained by seeking a solution to the governing PDE of the form of equation (11). In general, the exponents must be determined numerically, and the appropriate form of equation (11) obtained initially by guesswork.

In the following section, we consider a specific example with asymptotics of the second kind, and show that it is possible to determine systematically, using the RG, how many exponents are introduced, what their values are, and the form of the scaling function.

4. BARENBLATT'S EQUATION

When an elastic fluid flows through a porous medium which can expand and contract irreversibly, in response to the pressure u(x,t), the time evolution depends upon whether or not the pressure is increasing (medium expanding) or decreasing (medium contracting). The resulting equation for the pressure, using Darcy's Law, can be written as[1]

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \qquad (12)$$

with $D = \frac{1}{2}$ for $\partial_x^2 u > 0$ and $D = \frac{1}{2}(1+\varepsilon)$ for $\partial_x^2 u < 0$. The parameter ε depends upon the elastic constants of the fluid and the porous medium and the permeability. We consider only the initial value problem with $u \to 0$ as $|x| \to \infty$ and u(x,0) given by equation (4). The question we address is: what is the long-time behaviour of the Barenblatt equation?

The long-time behaviour cannot be of the form $u(x,t) \sim t^{-1/2} f(x\sqrt{t}, \varepsilon)$ for f having continuous second derivative. This can be seen by substituting this form into the Barenblatt equation; it is impossible to match across the point where $\partial_x^2 u = 0$. Nevertheless, uniqueness and existence of the initial value problem with continuous second derivatives in space have been proved.[7]

The renormalisation group approach to this initial value problem has six steps, each a direct counterpart of the procedure followed in quantum or statistical field theory. The first step is to construct a naive perturbation expansion in ε: it has the form, in the limit $\ell^2/t \to 0$,

$$u(x,t) \sim u_B(x,t) = \frac{A_0 e^{-x^2/2t}}{(2\pi t)^{1/2}} \left\{ 1 - \frac{\varepsilon}{(2\pi e)^{1/2}} \log \frac{t}{\ell^2} + O(\varepsilon^2) \right\} + \text{r.t.} \qquad (13)$$

where "r.t." stands for terms which are regular in the limit, and the subscript "B" stands for "bare", in conformity with field theoretic usage. It is convenient to achieve the limit by keeping t fixed, and letting $\ell \to 0$, as we did for the diffusion equation.

The second step is to cure the logarithmic divergence of the perturbation series, by introducing the renormalised pressure

$$u_R\,(x,t) = Z(\ell/\mu)\, u_B(x,t), \tag{14}$$

where μ is an arbitrary length, about which we will say more shortly. The renormalised pressure, u_R, will eventually be found to be the correct asymptotic solution of the Barenblatt equation, as opposed to the naive perturbation expression u_B which is (incorrectly) divergent. The function Z is referred to as a renormalisation constant, and strictly speaking, it is associated with $A_0 = \int u(x,0)dx$. Since the Barenblatt equation does not conserve $\int u(x,t)dx$, A_0 cannot be deduced from knowledge of u(x,t) at long times (i.e. when the origin of time is indeterminate). In this sense, A_0 is unobservable at long times, in the same way that the bare electric charge is unobservable at long distances, according to quantum electrodynamics. The renormalisation constant depends on ℓ, so that as $\ell \to 0$, the divergence in u_B may be absorbed into Z to yield a finite u_R. In the procedure described below, the removal of the divergence in u_B occurs order by order in ε, so we will assume that Z has an expansion in powers of ε. However, Z is, by definition dimensionless, and therefore cannot depend solely on the dimensional parameter ℓ. For this reason, an arbitrary parameter μ with the dimensions of length must be introduced.

In the third step of the renormalisation procedure, we expand

$$Z = 1 + a_1(\ell/\mu)\,\varepsilon + a_2(\ell/\mu)\varepsilon^2 + \ldots \tag{15}$$

and choose $a_1, a_2, \ldots$ to cancel order by order in ε the divergence in u_B as $\ell \to 0$. We find that

$$a_1 = \frac{1}{\sqrt{2\pi e}} \log\left[C_1(\mu^2/\ell^2)\right] \tag{16}$$

giving

$$u_R(x,t) = \frac{A_0 e^{-x^2/2t}}{\sqrt{2\pi t}} \left[1 - \frac{\varepsilon}{\sqrt{2\pi e}} \log\left(\frac{t}{C_1\mu^2}\right) + O(\varepsilon^2)\right] . \tag{17}$$

Here C_1 is an arbitrary number. Although this formula has two arbitrary parameters, it is more useful than it may seem; and it has the obvious virtue of being (trivially) finite as $\ell \to 0$, since ℓ is not present in the formula.

In fact, the formula (17) describes a family of solutions. Step four of the procedure is to chose a particular member of the family by requiring that (e.g.) at some time t*, the value of u_R at the origin is some number Q:

$$u_R(0,t^*) = Q\,.$$

Then the corresponding solution, to $0(\varepsilon)$ is

$$u_R(x,t) = Q\left(\frac{t^*}{t}\right)^{1/2} e^{-x^2/2t}\left(1 - \frac{\varepsilon}{\sqrt{2\pi e}}\log\frac{t}{t^*} + O(\varepsilon^2)\right). \tag{18}$$

This expression will be referred to as the renormalised perturbation expansion. Note that to this order in ε, the constants C_1 and μ have dropped out. A proof that this occurs to all orders in ε for the arbitrary constants C_1, C_2, ... introduced by the renormalisation procedure would constitute a proof of renormalisability. Although we do not doubt that the Barenblatt equation is renormalisable, we have not proven this. The renormalised perturbation expansion is useful at best only for times such that $1 \gg \frac{\varepsilon}{(2\pi e)^{1/2}}\log(t/t^*)$. For $t \gg t^*$, the renormalised perturbation expansion breaks down. Nevertheless, we will now show that the arbitrariness of t^* enables the renormalised perturbation expansion to be improved.

In step 5, we use the renormalisation group argument, due to Gell-Mann and Low[8]: The renormalised perturbation expansion involves a parameter t^* not present in the original problem. How can the asymptotics depend upon such a parameter? The point is that Q must also depend upon t^*. After all, during the diffusion process, $u(0,t)$ is expected to be a decreasing function of time. The dependence of Q on t^* can be found because its t^*-dependence must be that to cancel out the explicit t^*-dependence of the renormalised perturbation expansion. Thus

$$\frac{du_R}{dt^*} = \frac{\partial u_R}{\partial t^*} + \frac{\partial u_R}{\partial Q}\frac{dQ}{dt^*} = 0 . \tag{19}$$

The partial derivatives can be explicitly evaluated, at least to $0(\varepsilon)$, from equation (17). The result is

$$t^*\frac{dQ}{dt^*} = -Q\left[\frac{1}{2} + \frac{\varepsilon}{(2\pi e)^{1/2}} + 0(\varepsilon^2)\right] . \tag{20}$$

The final step is to solve this differential equation for Q. Substituting back into equation (17) and setting $t^* = t$, we finally obtain

$$u_R(x,t) = \frac{A}{t^{\alpha+1/2}}\, e^{-x^2/2t}\,(1 + 0(\varepsilon^2)) \tag{21}$$

with

$$\alpha = \frac{\varepsilon}{(2\pi e)^{1/2}} + O(\varepsilon^2). \tag{22}$$

Thus, the logarithmic terms in the perturbation expansion came from the expansion of $t^{-\alpha}$: the divergence of the perturbation series pointed the way to the correct asymptotics. The expansion of $\alpha(\varepsilon)$ is almost certainly divergent; it would be useful to know if it is Borel summable. We shall refer to α as the anomalous dimension.

5. GEOMETRICAL INTERPRETATION OF THE RG

So far, the exposition has been tied to perturbation theory. In fact, the RG approach is non-perturbative, and can be used as a procedure even when the expansion in ε is a poor approximation. We shall illustrate this here. Define the RG transformation on the space of functions $u(x,t_o)$ at a given value of t_o:

$$u'(x,t_o) \equiv R_{b,\phi}\left[u(x,t_o)\right]. \tag{23}$$

The RG transformation depends on 2 parameters and involves 3 steps: (1) Evolve the function $u(x,t_o)$ forward in time to $t_1 = bt_o$, $b > 1$, using the PDE. (2) Rescale x i.e. $x \to b^{\phi}x$, ϕ arbitrary. (3) Rescale the function itself so that $u'(0,t_o) = u(0,t_o)$. The general idea is that similarity solutions, if they exist, are fixed points of the RG transformation. That is, we iterate the RG transformation and if the initial conditions are in the basin of attraction of a fixed point, then the fixed point will be reached after an infinite number of iterations. The real power of the RG derives from the fact that it is relatively easy to approximate step (1), because the evolution is over a finite time. The RG procedure then is capable of giving a good approximation to the long time behaviour.

It should also be noted that for generic values of ϕ, there may not exist fixed points of $R_{b,\phi}$. Thus, ϕ must be varied to search for fixed points. To illustrate the RG expressed in this form, we will use the renormalised perturbation expansion to approximate step (1), again starting from a Gaussian. We obtain

$$u'(x,t_o) = \frac{1}{Z(b)}\, u(b^{\phi}x, bt_o) \tag{24}$$

with

$$Z(b) = b^{-1/2}\left(1 - \frac{\varepsilon}{(2\pi e)^{1/2}}\log b\right) + 0(\varepsilon^2) \tag{25}$$

The RG transformation forms a semi-group (semi, because $b > 1$):

$$R_{b_1,\phi}\; R_{b_2,\phi} = R_{(b_1 b_2),\phi}\,. \tag{26}$$

This implies $Z(b) = b^{y_Q}$ for any exponent y_Q, or

$$y_Q = \frac{\partial(\log Z)}{\partial(\log b)} = -\left(\frac{1}{2} + \frac{\varepsilon}{\sqrt{2\pi e}} + O(\varepsilon^2)\right) \tag{27}$$

Now let us determine ϕ. Performing one iteration on the initial condition, we obtain

$$u(b^{\phi}x, bt_o) = Q(t_o)\, b^{-1/2} \exp\left[-\, b^{2\phi}x^2/2bt_o\right] \times \left[1 - \frac{\varepsilon}{\sqrt{2\pi e}} \log b + O(\varepsilon^2)\right] \quad (28)$$

A fixed point is only possible if $\phi = 1/2$. At the fixed point

$$u^*(x,t) = b^{-y_Q}\, u^*(xb^{1/2}, bt) \ . \quad (29)$$

Choosing $b = 1/t$, we obtain the result.

$$u^*\,(x,\, t) = t^{-\alpha-1/2}\, u^*\!\left(\frac{x}{\sqrt{t}},\, 1\right) \quad (30)$$

with $\alpha(\varepsilon)$ given as in the preceding section.

We conclude this section with several remarks. First, step (1) can in principle be carried out numerically[9]. There is no restriction to using perturbation theory, as we have done here for pedagogical purposes. Second, the origin of the anomalous dimension in PDE problems is precisely the same as in critical phenomena. Consider, for example, the two-point correlation function G(k), of a scalar field at the critical point, as a function of wavenumber k: conventionally, $G(k) \sim k^{-2+\eta}$ as $k \to 0$, and μ is an anomalous dimension. We assume, for concreteness, that the field is defined on the vertices of a regular lattice, with lattice spacing ℓ. It can be shown that G must have the dimensions of $(\text{length})^2$. How then, can it have the conventional form at the critical point? The answer is that even though the correlations in the system have infinite range, the lattice spacing ℓ is still important and cannot be neglected (i.e. set to zero). In fact, the correlation function is singular as $\ell \to 0$:

$$G(k) \sim \ell^{\eta}\, k^{-2+\eta} \ , \quad (31)$$

It is this singularity, combined with the necessity to respect dimensional analysis, which leads to the anomalous wavenumber dependence of G(k) at the critical point. Finally, the geometrical formulation of the RG given here is the counterpart of Wilson's method in statistical physics.[10]

6. SPECULATIONS

We conclude this article by offering a number of speculations[2] on future applications of this work. At the time of writing, these and other avenues of research are being actively followed.

Perhaps the most interesting application, from the point of view of this workshop, is to velocity selection in dendritic growth. We shall restrict ourselves to models such as the geometric model[11] (GM) or the boundary-layer model[12], where there is no doubt about the procedure to construct the steady states or needle crystals. In the GM, for example, the velocity of steady states is given by the non-linear eigenvalue problem

$$\left(\kappa + \frac{\partial^2\kappa}{ds^2}\right) = \frac{v \cos\theta}{1 + \varepsilon \cos(m\theta)} \tag{32}$$

with boundary conditions

$$\left.\frac{d\kappa}{ds}\right|_{s=0} = 0\,; \quad x \to 0 \text{ as } s \to \infty\,. \tag{33}$$

Here, $\kappa = d\theta/ds$ is the curvature of an interface whose normal is at an angle θ to the axis of symmetry of the needle crystal, at an arc length distance s from the tip. The degree of symmetry is m. Only for special values of v can the boundary conditions be satisfied. The solution corresponding to the largest value of v is the needle crystal which forms the tip of the dendrites in the GM. The steady state equation (32) is the analogue of the equation determining the scaling function f and the anomalous dimension α in Barenblatt's equation: substituting $u = t^{-(\alpha + 1/2)} f(\xi)$, $\xi = x/\sqrt{t}$ in Barenblatt's equation yields a non-linear eigenvalue equation for a and f. Indeed, this is typically how Barenblatt and others have solved problems with asymptotics of the second kind.

It is natural to conjecture that the scaling of v with ε can be obtained by studying the asymptotics of the initial value problem giving rise to (32), for small ε:

$$\left.\frac{\partial\kappa}{\partial t}\right)_\theta = -\left(\kappa^2 + \frac{\partial^2}{\partial\theta^2}\right)\left[1 + \varepsilon\cos(m\theta)\right]\left[\kappa + \frac{\partial^2\kappa}{\partial s^2}\right].$$

However, this is problematic because the time evolution for $\varepsilon = 0$ is quite different from that when $\varepsilon \neq 0$. In the Barenblatt equation, on the other hand, the expansion parameter ε had no qualitative effect on the time evolution--the perturbation is a marginal operator in the language of statistical mechanics. We do not yet know how to resolve this apparent difficulty.

Finally, it is of interest to apply the converse of our results to statistical mechanics: instead of determining critical exponents by performing successive renormalisation group transformations in space, which is the analogue of the initial value problem for PDEs, can one determine a non-linear eigenvalue problem for the critical exponents, which is the analogue of the steady state equation (or the equation for the scaling function f)?

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