Emergent States of Matter

HOMEWORK SHEET 7

Due 12 noon Wed 5 May 2021 or earlier

Please attempt these questions without looking at textbooks, if you can. You will learn more by thinking about these problems yourself.

Question 7–1.

This question concerns the behaviour of correlations in a system with a continuous symmetry at its lower critical dimension. This is the dimension above which ordering is possible for non-zero T. We saw earlier that for Bose-Einstein condensation, described by a complex order parameter, $d_c = 2$. Note that a similar more technically complicated analysis applies to smectic liquid crystals, where $d_c = 3$.

(a) Consider a system with a complex order parameter $\psi = S_1 + iS_2$ described by the Hamiltonian

$$-\mathcal{H} = \int d^2r \left[\frac{1}{2}|\nabla\psi|^2 + \frac{u_0}{4}\left(|\psi|^2 - \frac{|r_0|}{u_0}\right)^2 \right]$$

Notice that this is just a particular way to write our usual Landau theory up to quartic order, for a complex order parameter. At low temperatures, the amplitude degrees of freedom are frozen out, but the phase fluctuations are strong as we have seen previously. Writing $\psi = A\exp(i\theta(\mathbf{r}))$, where A is the temperature dependent amplitude, which you should determine, show that the effective Hamiltonian is

$$-\mathcal{H} = \frac{K}{2}\int d^2r(\nabla\theta)^2$$

and determine the spin wave stiffness K. The Hamiltonian we have found is that of a 2D XY spin system with effective exchange interaction $J = K(k_BT)$.

(b) Using the fact that the Hamiltonian is Gaussian, show that the normalized order parameter correlation function $G(\mathbf{r}) \equiv \langle\psi^*(\mathbf{r})\psi(0)\rangle / A^2$ obeys

$$G(\mathbf{r}) = \exp\left[-\frac{1}{2}\frac{2k_BT}{J}\int_0^\Lambda \frac{d^2k}{(2\pi)^2}\frac{1 - e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2} \right]$$

where Λ is the coarse-graining scale.

(c) For large $r \gg \Lambda^{-1}$, the oscillatory exponential term can be neglected. By change of variable, and making sure to consider what happens to the limits of integration when doing so, show that the integral can be approximated by its logarithmic divergence, leading to $G(r) = \left(\frac{r}{\Lambda^{-1}}\right)^{-\eta}$, where $\eta = k_BT/2\pi J$

This important result shows that at the lower critical dimension, correlations exhibit power law decay, with a temperature dependent, continuously varying exponent. At higher temperatures, our phase approximation must break down, and at higher temperatures still, we expect the system to exhibit the usual exponential decay of correlations. Hence we conclude that at some intermediate temperature, there is a phase transition between a state with exponential correlations and a state with power law correlations. This is the celebrated Kosterlitz-Thouless transition.

Question 7–2.

(a) Starting from the Hamiltonian for a 2D superfluid

$$\mathcal{H} = \frac{\rho_s}{2}\int v_s^2\, d^2\mathbf{r}$$

calculate the energy of a quantum vortex in a circular 2D condensate of radius R, in terms of the system radius R and core size a and C, the energy/unit area of the region of the core.

(b) Consider two quantum vortices in the 2D condensate, separated by a distance r, with topological charges $q = \pm 1$. Calculate the energy of a pair of vortices, one with sign $+1$, the other with sign -1, and show that

$$E(r) = 2\pi\rho_s(\hbar/m)^2 \ln(Cr/a)$$

Note that this energy is finite, so vortex pairs can be thermally excited for $T > 0$.

(c) **OPTIONAL - no credit** If the vortices are initially placed in an $x - y$ coordinate system with the $+1$ vortex at $y = r/2$ and the -1 vortex at $y = -r/2$, what is the resulting magnitude and direction of the velocity of the pair?

(d) Suppose now that there is only a single vortex in the system. Estimate the number of ways that the vortex can be placed in the system, and hence calculate the free energy of the single vortex. You should find that

$$F/k_BT = (\pi K - 2)\ln R + \cdots$$

where $K \equiv (\hbar/m)^2\rho_s/k_BT$. Hence show that as long as we can neglect the interaction between vortices, it is thermodynamically favorable for vortices to proliferate above at critical temperature given implicitly by $K^* = 2/\pi$. This is the Kosterlitz-Thouless transition temperature, T_{KT} and the result obtained here by elementary methods is correct when treated by renormalization group methods. Comment on whether or not you think that this transition has emerged through the mechanisms of spontaneous symmetry breaking that we have focused on in other systems during the course.

Note also that your result implies that going below T_{KT}, the superfluid density jumps from zero to a value that depends only on T_{KT} and fundamental constants, a prediction that was experimentally verified and cited in the Nobel Prize description for Kosterlitz and Thouless.

(e) Using this answer and your results for 7–1, show that correlations decay algebraically at the transition temperature with a 1/4 power law.

Question 7–3.

This question concerns spontaneous symmetry breaking in charged superfluids, i.e. superconductors, and is based on the coarse-grained free energy

$$F\{\psi, \vec{A}\} = \int d^dx \left\{ \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + |(\nabla - ie^* \vec{A})\psi|^2 + \frac{(\nabla\times \vec{A})^2}{8\pi} \right\}$$

where e^* is the effective charge of the condensate, which we know to be $e^* = 2e$ from BCS theory, α is proportional to $T - T_c$, ψ is the complex order parameter for the superconducting transition, and $\vec{A}$ is the electromagnetic vector potential. In the absence of an electromagnetic field, the order parameter spontaneously breaks the "U(1) global gauge symmetry" (i.e. the invariance to changes of phase of ψ) when $\alpha(T) < 0$ and $\psi^2 = -\alpha/\beta$. In the following we will chose the phase of ψ to be zero, so that $\psi = v \equiv \sqrt{-\alpha/\beta}$.

(a) Let $\vec{A}$ now be non-zero, and expand ψ about v to second order to find the effective free energy for the system in terms of the real and imaginary parts of the fluctuation in ψ, ψ_1 and ψ_2 respectively. It will be most convenient for you to calculate $\Delta F \equiv F\{\psi, \vec{A}\} - F\{\psi, \vec{0}\}$. Don't forget the gradient terms.

(b) Your resulting expression is difficult to interpret physically because it involves a cross term between a component of the order parameter and the vector potential. Show that this can be removed by making a gauge transformation on the vector potential: $\vec{A} = \vec{A}' + \nabla\Lambda$ where $\vec{A}'$ is the transformed vector potential and Λ is a function that you should determine.

(c) Your resulting expression for ΔF should contain ψ_1 and $\vec{A}'$ only. What happened to ψ_2? Has the number of degrees of freedom changed during the transition?

(d) Explain the physical significance of the $\vec{A}'$ dependence, by writing the Euler-Lagrange equations for ϕ_1 and $\vec{A}'$. Be sure to explain what is the "mass" of all the fields left in the problem. What happened to the Goldstone boson that is present in the neutral superfluid case when $e^* = 0$?

This phenomenon is known as the Anderson-Higgs mechanism and plays an important role in condensed matter physics and higher energy physics.