Bose-Einstein Condensation in Trapped Atomic Gases

Minqiang Li

Department of Physics, UIUC

Abstract

In this paper, I will first talk about ideal gases in trapped potentials, such as transition temperature and various temperature dependences of physics quantities. Then I will go on to talk in some detail the interacting bose gases. I will first use a mean field theory approach to study the ground state properties. The next topic is oscillation modes which can be observed experimentally. We will see that despite a dilute nature, the interaction changes the properties of the system significantly.

I. INTRODUCTION

In 1995, Bose-Einstein condensation (BEC) was first observed in a remarkable series of experiments on vapors of rubidium and sodium. This soon opened a rapidly developing area of both experimental and theoretical research. Now more than 10 groups have achieved Bose-Einstein condensation in various system, including hydrogen, lithium and more recently, in molecular bi-condensate systems.

The reason that Bose-Einstein condensation has only been achieved tens of years later than Einstein first published his paper perhaps is due to lack of sophisticated cooling techniques. In most of the successful experiments, the atoms were confined in magnetic traps and then cooled down to extremely low temperatures, of the order of fractions of microkelvins.

Accordingly, one of the most relevant features of these trapped gases is that they are inhomogeneous and finite-sized systems, the number of atoms ranging typically from a few thousand (attractive interacting systems, such as ^{7}Li) to several million. A lot of analytical work can still be done by approximating the confining traps as harmonic potentials. A characteristic length scale can be formed using the trapping frequency ω_{ho}, $a_{ho} = [\hbar/(m\omega_{ho})]^{1/2}$, which is of the order of a few microns in most cases. Density variations occur on this scale.

Several important consequences arise from this inhomogeneity. First, BEC shows both condensation in momentum space and real coordinate space. This actually turns out to be the heaven for both theorists and experimentalists because theoretical predictions can often be verified by investigations in interference phenomena, collective modes, etc. Another important consequence is that the two-body interactions between atoms, which typically play little role when the gases are dilute, are greatly enhanced as seen from both theoretical work and experiments on many measurable quantities. Nontheless, the dilute nature will turn out to be helpful; in most cases, a single parameter, the s-wave scattering length, is sufficient to obtain accurate descriptions.

In this paper, we will first study in a little bit detail ideal bose gases in a harmonic trap. We will discuss the energy density and the thermodynamics in 'the thermodynamic limit' plus some relevant remarks. Then we will loose the constraint of ideal gases and study interacting bose gases in the presence of confining potentials. We will study in detail the ground state, the collective oscillation modes using a mean field approach, which in weakly interating systems takes the form of Gross-Pitaevskii theory. Because of limit of space, a lot of intersting topics, such as phase coherence phenomena, will not be discussed here. Readers interested should refer to the references cited here.

II. THE IDEAL BOSE GAS IN A HARMONIC TRAP

We start by considering a confining potential of the following quadratic form

$$V_{ext}(\mathbf{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)\,. \tag{1}$$

In our first step, we will ignore atom-atom interaction. In this case, a lot of analytical work can be done. The many-body Hamiltonian simplifies to the sum of single-particle Hamiltonians whose eigenvalues have the form:

$$\epsilon_{n_x n_y n_z} = (n_x + \frac{1}{2})\hbar\omega_x + (n_y + \frac{1}{2})\hbar\omega_y + (n_z + \frac{1}{2})\hbar\omega_z\ , \tag{2}$$

where $\{n_x, n_y, n_z\}$ are non-negative integers. The ground state is obtained by putting all the particles in the lowest single-particle state ($n_x = n_y = n_z = 0$), namely $\psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \prod_i \phi_0(\mathbf{r}_i)$, where $\phi_0(\mathbf{r})$ is given by

$$\phi_0(\mathbf{r}) = (\frac{m\omega_{ho}}{\pi\hbar})^{3/4} \exp[-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)]\,. \tag{3}$$

The density distribution becomes $n(\mathbf{r}) = N|\phi_0(\mathbf{r})|^2$. The size of the cloud can be seen to be

$$a_{ho} = (\frac{\hbar}{m\omega_{ho}})^{1/2}\,, \tag{4}$$

where ω_{ho} is the geometric average of the oscillator frequencies:

$$\omega_{ho} = (\omega_x\omega_y\omega_z)^{1/3}\,. \tag{5}$$

This discussion shows that there is a sharp peak in the central region of the density distribution. Indeed, experiments on column density confirmed this prediction. Of course, there are discrepancies between the two because we have not included two-body interaction effects in our discussion.

Another interesting thing is when we use axially symmetric traps. The ratio between the axial and radial frequencies is $\lambda = \omega_z/\omega_\perp$. A similar arguments as before will tell us that now the shape of the expanded cloud in the x-z plane is an ellipse, with the ratio between the two axis (which is called the aspect ratio) equal to $\sqrt{\lambda}$. This is very important experimentally. If there is no condensate, then equipartition theorem will tell us that the aspect ration should be equal to 1. So the anisotropic distribution gives an important signature of BEC.

Now we go on to discuss the ideal bose gases at finite temperatures. In the grand canonical ensemble, the total number of particles and the total energy are given by

$$N = \sum_{n_x,n_y,n_z} \{\exp[\beta(\epsilon_{n_x n_y n_z} - \mu)] - 1\}^{-1} \tag{6}$$

$$E = \sum_{n_x,n_y,n_z} \epsilon_{n_x n_y n_z} \{\exp[\beta(\epsilon_{n_x n_y n_z} - \mu)] - 1\}^{-1} \tag{7}$$

Denote the paricle number in the ground state N_0, we then get

$$N - N_0 = \sum_{n_x,n_y,n_z \neq 0} \frac{1}{\exp[\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)] - 1} \,. \tag{8}$$

When the chemical potential becomes equal to the energy of the lowest state,

$$\mu \longrightarrow \mu_c = \frac{3}{2}\hbar\bar{\omega} \,, \tag{9}$$

where $\bar{\omega} = (\omega_x + \omega_y + \omega_z)/3$ is the arithmetic average of the trapping frequencies, this N_0 can be macroscopic. If the number of particles N is large and $k_B T \gg \hbar\omega_{ho}$, then we can well approximate the sum by an integral. The result is readily obtained using the Riemann ζ-function to be

$$N - N_0 = \zeta(3)(\frac{k_B T}{\hbar\omega_{ho}})^3 \,. \tag{10}$$

Now imposing the condition that $N_0 \to 0$ at the transition T_c^0, we get

$$k_B T_c^0 = \hbar\omega_{ho}(\frac{N}{\zeta(3)})^{1/3} \,. \tag{11}$$

We can take the proper thermodynamic limit by letting $N \to \infty$ and $\omega_{ho} \to 0$, while keeping the product $N\omega_{ho}^3$ constant. Thus the transition temperature is well defined in the thermodynamic limit. From the above two equations, we readily have the following temperature dependence for $T < T_c^0$:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c^0}\right)^3 \,. \tag{12}$$

A similar relation for the energy can also be obtained to yield the following:

$$\frac{E}{Nk_BT_c^0} = \frac{3\zeta(4)}{\zeta(3)}\Big(\frac{T}{T_c^0}\Big)^4 . \tag{13}$$

We can go on to calculate the specific heat, entropy and other thermodynamic quantities starting from the energy.

These results, as we see, yield quite different temperature behavior from the uniform case, whose familiar relations are listed below for a comparison:

$$\frac{N_0}{N} = 1 - \Big(\frac{T}{T_c^0}\Big)^{3/2} , \tag{14}$$

$$k_BT_c^0 = \frac{2\pi\hbar^2}{m}\Big[\frac{n}{\zeta(3/2)}\Big]^{2/3} , \tag{15}$$

$$\frac{E}{Nk_BT_c^0} = \frac{3\zeta(5/2)}{2\zeta(3/2)}\Big(\frac{T}{T_c^0}\Big)^{5/2} . \tag{16}$$

The approximations we made concern with two energies scales: $k_BT_c^0$ and $\hbar\omega_{ho}$. In most of the traps, the former is about 2 orders of magnitude greater than the latter, so semiclassical approximation is expected to work well in these systems in a wide range of temperatures. In fact, the measured transition temperatures were observed to be very close to the ideal gas value given above. A more accurate comparison can be done when we consider finite size effects and the two-body interaction, etc. For example, the finite size correction can be evaluated by studying the large N limit of the sum instead of an integral and the result for N_0/N is given by [1]

$$\frac{N_0}{N} = 1 - \Big(\frac{T}{T_c^0}\Big)^3 - \frac{3\bar{\omega}\zeta(2)}{2\omega_{ho}[\zeta(3)]^{2/3}}\Big(\frac{T}{T_c^0}\Big)^2 N^{-1/3} . \tag{17}$$

Before we go on to interacting bose systems, let's briefly mention ideal bose gases in nonharmonic traps and the possibility of cooling by adiabatic transformations (by which we mean changing the shape of the trap at a rate slow compared to the internal equilibrium rate). For example, suppose we have a potential given by the power-law form:

$$V_{ext}(\mathbf{r}) = Ar^{3/\delta}\,. \tag{18}$$

Above T_c, the entropy per particle takes the form of [2]

$$\frac{S}{Nk_B} = \Big(\frac{5}{2} + \delta - \ln\zeta(3/2+\delta)\Big) + (3/2+\delta)\ln(\frac{T}{T_c^0})\,. \tag{19}$$

From the above expression, we see that if we increasing δ adiabatically, for example, in going from a harmonic trap ($\delta = 3/2$) to a linear one ($\delta = 3$), one observes a reduction of the ratio $t = T/T_c^0$:

$$t_2 \simeq 0.7 t_1^{2/3}\,. \tag{20}$$

In this case, a system at twice the critical temperature ($t_1 = 2$) can be cooled down to nearly the critical temperature ($t_2 \approx 1.1$). Using this technique, it should be possible to cool a system adiabatically down to temperatures below T_c.

III. INTERACTING BOSE GASES

We start by writing out the many body Hamiltonian describing N interacting bosons with an external potential V_{ext}:

$$\hat{H} = \int d\mathbf{r}\,\hat{\Psi}^\dagger(\mathbf{r})\Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r})\Big]\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\,\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r})\,, \tag{21}$$

where $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})$ are the boson field annihilation and creation operators and $V(\mathbf{r}-\mathbf{r})$ is the two-body interatomic potential.

In the case of nonuniform bose gases, we can still use a procedure first formulated by Bogoliubov. The central idea to is separate out the condensate contribution to the bosonic field operator. In general, our field operator can be written as $\hat{\Psi}(\mathbf{r}) = \sum_\alpha \Psi_\alpha(\mathbf{r})a_\alpha$, where Ψ_α are single particle wave functions and a_α are the corresponding annihilation operators. Bose-Einstein condensation occurs when the number of atoms N_0 becomes macroscopically large, so that we can treat the operators a_0 and $a_0^\dagger$ as c-numbers: $a_0 = a_0^\dagger = \sqrt{N_0}$. The field operator written in Heisenberg representation can be decomposed into the form

$$\hat{\Psi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \hat{\Psi}'(\mathbf{r},t) \tag{22}$$

where the condensate density is given by

$$n_0(\mathbf{r},t) = |\Phi(\mathbf{r},t)|^2 \,. \tag{23}$$

and we can treat $\hat{\Psi}'(\mathbf{r},t)$ as small perturbation.

By approximating $V(\mathbf{r}' - \mathbf{r}) = g\delta(\mathbf{r}' - \mathbf{r})$ and using the Heisenberg equation of motion, we get the following closed equation for the order parameter when neglecting the thermal components:

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\Big)\Phi(\mathbf{r},t)\,, \tag{24}$$

where the coupling constant g is related to the s-wave scattering length a through

$$g = \frac{4\pi\hbar^2 a}{m}\,. \tag{25}$$

This equation is called the Gross-Pitaevskii equation, which can also be derived using a variational procedure if we take the energy functional to be

$$E[\Phi] = \int d\mathbf{r}\,\Big[\frac{\hbar^2}{2m}|\nabla\Phi|^2 + V_{ext}(\mathbf{r})|\Phi|^2 + \frac{g}{2}|\Phi|^4\Big]\,. \tag{26}$$

The first term in the integral is the kinetic energy of the condensate E_{kin}, the second is the harmonic oscillator energy E_{ho}, while the last one is the mean field interaction energy E_{int}. By dilute gases we mean that $\bar{n}|a|^3 \ll 1$, where $\bar{n}$ is the average density of the gas. E_{int} is given by $gN\bar{n}$ and the average density is of the order of N/a_{ho}^3. On the other hand, the kinetic energy is of the order of $N\hbar\omega_{ho}$. So we see that

$$\frac{E_{int}}{E_{kin}} \propto \frac{N|a|}{a_{ho}}\,. \tag{27}$$

This parameter has great importance when we make various approximations. Because we have neglected the thermal component, our formalism is only strictly valid at zero temperature. For a treatment in finite temperatue, see [7].

A. Ground state

The ground state can be easily obtained within the framework of mean field theory. We write $\Phi(\mathbf{r},t) = \phi(\mathbf{r})\exp(-i\mu t/\hbar)$, where μ is the chemical potential and ϕ is real and normalized to the total number of particles. The Gross-Pitaevskii equation becomes

$$\Big(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{r}) + g\phi^2(\mathbf{r})\Big)\,\phi(\mathbf{r}) = \mu\phi(\mathbf{r})\,, \tag{28}$$

which is sometimes called the "nonlinear Schrödinger equation". This equation can be solved numerically to give the density profile which we will not discuss here. Instead, we will derive an exact virial theorem which can be used to check numerical simulations.

For the ground state, since there is no current, the energy functional can be written only of the density:

$$E[n] = \int d\mathbf{r}\,\Big[\frac{\hbar^2}{2m}|\nabla\sqrt{n}|^2 + nV_{ext}(\mathbf{r}) + \frac{gn^2}{2}\Big] = E_{kin} + E_{ho} + E_{int}\,. \tag{29}$$

By directly integrating the Gross-Pitaevskii equation, one gets

$$\mu = (E_{kin} + E_{ho} + 2E_{int})/N \tag{30}$$

Further relationships can be found by means of virial theorem. The energy functional is stationary for any variations of ϕ around the exact solution of the GP equation. In particular, we can choose a scaling transformations of the form

$$\phi(x,y,x) \longrightarrow (1+\nu)^{1/2}\phi((1+\nu)x,y,z)\,. \tag{31}$$

By imposing the variation to vanish at first order in ν, we can get

$$(E_{kin})_x + \frac{1}{2}E_{int} - (E_{ho})_x = 0\,, \tag{32}$$

where $(E_{ho})_x = \frac{m\omega_x^2}{2}\langle\sum_i x_i^2\rangle$ and $(E_{kin})_x = \langle\sum_i p_{ix}^2\rangle/2m$. Similar expressions can be got for the y and z co-ordinates. Summing over the three directions we get a beautiful exact result for the ground state:

$$2E_{kin} - 2E_{ho} + 3E_{int} = 0\,. \tag{33}$$

In a series of experiments, the gas has been imaged after a sudden switching-off of the trap. The cloud then expand with constant release energy which is the sum of kinetic energy and interaction energy. This can be measured experimentally and provides a direct comparison between theory and experiments. The two agreed pretty well. [9]

Next we consider two cases of interactions: attractive and repulsive. For systems such as ^{7}Li, $a < 0$, so the gas tends to increase its density in the center of the trap in order to lower the interaction energy. It has been showed that there exists a critical N_{cr}; above this value there no longer exists a minimum of the energy functional and the Gross-Pitaevskii equation has no solution. This N_{cr} can be analyzed to give a value close to 1400 for ^{7}Li systems used in the Rice group. The interesting collapse when N is large has been studied using different approaches. Ueda and Leggett studied the role played by quantum tunneling phenomena. [3]

In the case of repulsive forces and large N, one can use a Thomas-Fermi approximation by neglecting the quantum pressure term $\nabla^2\sqrt{n}$. One then finds

$$n(\mathbf{r}) = \phi^2(\mathbf{r}) = g^{-1}[\mu - V_{ext}(\mathbf{r})] \tag{34}$$

in the region where $\mu > V_{ext}(\mathbf{r})$, and $n = 0$ outside. The normaliation condition for n gives the following relation:

$$\mu = \mu_{TF} = \frac{\hbar\omega_{ho}}{2}\Big(\frac{15Na}{a_{ho}}\Big)^{2/5}\,. \tag{35}$$

Since $\mu = \partial E/\partial N$, the energy per particle turns out to be [4]

$$\frac{E}{N} = \frac{5}{7}\mu, \tag{36}$$

which should become more and more accurate as $N \to \infty$.

The turning point $\mathbf{R}$ is defined by the condition $\mu = V_{ext}(\mathbf{R})$. For a spherical trap, we get the following:

$$R = a_{ho}\Big(\frac{15Na}{a_{ho}}\Big)^{1/5} , \tag{37}$$

which grows with N. The Thomas-Fermi approximation is bad around the turning point because spatial fluctuation here is large. An improved approximation also takes into account the density fluctuation close to the turning point. In fact, when $|r - R| \ll R$, the trapping potential V_{ext} can be replaced by a linear ramp, $m\omega_{ho}^2 R(r - R)$, which then can be solved analytically. [4] I will just quote the results here:

$$\mu = \mu_{TF}\Big[1 + 3\frac{a_{ho}^4}{R^4}\ln(R/Ca_{ho})\Big] , \tag{38}$$

and

$$E = \frac{5}{7}N\mu_{TF}\Big[1 + 7\frac{a_{ho}^4}{R^4}\ln(R/Ca_{ho})\Big] , \tag{39}$$

where $C \simeq 1.3$ is a numerical factor. When N is large, the Thomas-Fermi approximation is sometimes indistinguishable from exact numerical solutions of the Gross-Pitaevskii equation. As we will see, the Thomas-Fermi approximation is also very useful when we study the dynamics and thermodynamics of the system. Here by dynamics we mean excitation modes due to interactions, large amplitude oscillations, hydrodynamics, etc. and by thermodynamics we mean finite temperature properties of the system.

Another interesting thing is that the Thomas-Fermi result can be compared with variational calculations. In the absence of interparticle interactions the lowest single-particle state has the familiar wave function:

$$\phi_0(\mathbf{r}) = \frac{1}{\pi^{3/4}(a_1 a_2 a_3)^{1/2}} \exp(-x^2/2a_1^2)\exp(-y^2/2a_2^2)\exp(-z^2/2a_3^2) , \tag{40}$$

where a_i are oscillator lengths given by $a_i^2 = \hbar/m\omega_i$. We then loose the constraints on a_i and treat them as variational parameters and use the above wavefunction as a trial function. A typical variational calculation gives

$$\frac{E}{N} = \frac{5}{4}(\frac{2}{\pi})^{1/5}\beta^2\hbar\bar{\omega} , \tag{41}$$

which gives a result very close to the Thomas-Fermi approximation. (see equations (35), (36).)

Now before closing this discussion on the ground state properties, we will talk about the validity of mean field theory approach. If the system is dilute, the results should be pretty accurate. Finite value of N will certainly provides corrections to our result, which can be studied in the framework of local density approximation, etc. For example, one possible correction is to include higher orders in the Bogoliubov approximation, which was pointed out by Yang and Lee to be

$$\mu = gn\Big[1 + \frac{32}{3\sqrt{\pi}}(na^3)^{1/2}\Big]\,. \tag{42}$$

For a detailed discussion, readers are referred to some recent papers cited here. [5] [6]

B. Dynamics and excitations

In this section, we will study the collective modes due to interactions. Most of our result concerns with zero temperature. Readers can read the papers cited at the end of [7] if interested in finite temperature effects.

We start by citing the elementary excitations in uniform bose gases. Using Bogoliubov transformation, it is not difficult to obtain

$$E_k = \sqrt{(\epsilon_k + NU_0/V)^2 - (NU_0/V)^2}\,, \tag{43}$$

and we can interprete it as the dispersion relation of a quasiparticle. In the case of nonuniform systems, the collective modes we will obtain will not have such an important role. [7] But still it is important to derive them since it gives direct comparison with experimental results. We will use an axially symmetric potential $V(\mathbf{r})$ of the following form for illustration on how one can generally obtain such frequencies:

$$V(x,y,z) = \frac{1}{2}m\omega_0^2(\rho^2 + \lambda^2 z^2)\,, \tag{44}$$

where $\rho^2 = x^2 + y^2$. We rewrite the time-dependent Gross-Pitaevskii equation below:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t) + U_0|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t) = i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}\,. \tag{45}$$

By using the complex conjugate of the above equation, we can get a continuity equation of the following form:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\,, \tag{46}$$

where the velocity is given as always:

$$\mathbf{v} = \hbar\frac{\psi^*\nabla\psi - \psi\nabla\psi^*}{2mi|\psi|^2}\,. \tag{47}$$

If we rewrite the wavefunction in terms of its module and phase,

$$\psi = \sqrt{n}e^{i\phi}\,, \tag{48}$$

then we obtain the following equation of motion for the velocity field:

$$m\frac{\partial\mathbf{v}}{\partial t} = -\nabla(\delta\mu + \frac{1}{2}mv^2)\,, \tag{49}$$

where

$$\delta\mu = V + U_0 n - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n} - \mu_0\,. \tag{50}$$

The equation $\delta\mu = 0$ can be seen to be identical to the time-independent Gross-Pitaevskii equation.

Within the Thomas-Fermi approximation, we can write the equilibrium chemical potential by neglecting the kinetic energy:

$$n_0 U_0 + V(x,y,z) = \mu_0\,. \tag{51}$$

In the spirit of the Thomas-Fermi approximation, we can also neglect the kinetic energy term involving δn in the expression for $\delta\mu$ and thus obtain

$$\delta\mu = U_0 \delta n \,. \tag{52}$$

We then linearize the continuity equation and the equation of motion for the velocity field as always and eliminate $\delta\mu$ by means of the above relation. The result is

$$m\frac{\partial^2 \delta n}{\partial t^2} = U_0 \nabla \cdot (n_0 \nabla \delta n) \,. \tag{53}$$

If we only consider oscillations with the time dependence $\delta n \propto \exp(-i\omega t)$, then the above differential equation simplifies into the following three equations

$$-\omega^2 \delta n = \frac{U_0}{m}(\nabla n_0 \cdot \nabla \delta n + n_0 \nabla^2 \delta n) \,, \tag{54}$$

$$n_0 = \frac{\mu_0}{U_0}\Big(1 - \frac{\rho^2}{R^2} - \frac{\lambda^2 z^2}{R^2}\Big) \,, \tag{55}$$

where the turning radius is give by $V(\mathbf{R}) = 0$,

$$R^2 = \frac{2\mu_0}{m\omega_0^2} \,. \tag{56}$$

Because of the axial symmetry, there are solutions proportional to $\exp(im\phi)$, where m is an integer. Thus we can try the following form of solution

$$\delta n \propto r^l Y_{lm}(\theta, \phi) \,. \tag{57}$$

When $m = \pm l$, $\nabla^2 \delta n = 0$, and we see the frequencies are given by $\omega^2 = l\omega_0^2$. Likewise there are solutions of the above form with $m = \pm(l-1)$.

In the following we investigate the low-lying modes which have experimental significance. We try a solution of the form ($m = 0$)

$$\delta n = a + b\rho^2 + cz^2 \,, \tag{58}$$

where a, b, c are constants to be determined. Upon insertion of this trial solution into the linearized equations, we obtain three algebraic equations for a, b and c. The condition that non-trivial solution exists is

$$(\omega^2 - 4\omega_0^2)(\omega^2 - 3\lambda^2\omega_0^2) - 2\lambda^2\omega_0^4 = 0\,, \tag{59}$$

which for the typical aspect ratio value of $\lambda^2 = 8$, gives

$$\omega = \omega_0(14 - 2\sqrt{29})^{1/2} \approx 1.797\omega_0\,. \tag{60}$$

Both this $m = 0$ mode and the $m = 2$ mode with frequency $\sqrt{2}\omega_0$ have been observed experimentally. [8]

IV. CONCLUSIONS AND OUTLOOK

After the realization of the first BEC, it has grown to be an area of vast research interest. One reason is that it provides a direct testground for many body theories. Another reason remains in its potential application, for example, quantum computation, becuase of both its high density and phase coherence properties (which we do not have a chance to talk about here). We have seen that a mean field theory approach, which takes the form of Gross-Pitaevskii equation and Thomas-Fermi approximation, describes pretty well the behavior of the condensate. Areas that are now being studied intensely on include the following: trapped Fermi gases, mixture of condensates, optical confinement, etc.

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