Kosterlitz-Thouless Transition in Superfluid Helium Films

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Abstract

Superfluid flow in thin films was first observed in 1950, contradicting theoretical works that had predicted the destruction of the superfluid order in two dimensions.

Experimental data supports the theory that the superfluid phase in thin helium films is realised through the Kosterlitz-Thouless two-dimensional phase transition. However, this low temperature phase does not exhibit a true long range order as that in bulk samples, but instead, a power-law type decay of the correlation length is observed. Interestingly enough, despite this quasi long range order, these films can still exhibit superfluidity.

In Kosterlitz-Thouless theory of transition in superfluid films, the disappearance of superfluidity is mediated by thermally excited vortices. At low temperatures, it is energetically favourable for vortices to form vortex-antivortex pairs. At a sufficiently high temperature, these vortex pairs start unbinding. At this critical temperature, correlation length has only a short range order characteristic of a disordered state and, apparently, supefluid flow is no longer supported. As a direct consequence of this, the superfluid areal density vanishes.

The most important experimental techniques are described as well.

1 Introduction

The possibility of superfluid phase transition at non-zero temperature in structures with dimensionality $d < 3$ is strictly excluded by rigorous theoretical proofs which show that the superfluid order parameter is destroyed in two or fewer dimensions, and hence superfluid flow cannot be realised.

In spite of this, superfluidity in thin films was observed in 1950 by Long and Meyer. Shortly after this startling discovery many other experiments followed and a strong body of evidence was formed in favour of the seemingly contradicting fact that superfluidity in $d < 3$ structures can be observed at $T > 0$.

Many theories have been proposed in order to explain this behaviour of thin films, but the most successful theory so far, is the theory of J.M. Kosterilitz and D.J. Thouless [1, 2].

It correclty predicts many features of supefliud helium films, among which the most important result is that the ratio of the density per unit area of the superfluid component of a helium film to the temperature at the critical point (the onset of superfluidity) as $T \to T_c$, is a universal constant:

$$\frac{\bar{\rho_{sf}} d}{T_c} = \frac{2}{\pi}\left(\frac{m}{\hbar}\right)^2 k_B \tag{1}$$

Nelson and Kosterlitz even suggested that this result could be a criterion of validity of various theories that address this phenomenon.

Acoording to this theory, the superfluid phase transition in thin ^{4}He films is realised through a Kosterlitz-Thouless two-dimensional phase transition. Unlike bulk helium, this low-temperature phase dose not exhibit a true long range order, but instead, the order parameter correlation length decays according to a power-law.

Although a true long range order is absent, this quasi-long range order still allows for these films to exhibit superfluidity.

The mechanism that contributes the most to the dissapearance of superfludity is the thermal excitation of vortices. In the Kosterlitz-Thouless theory, the two-dimensional superfluid at low temperatures is populated by a system of bound vortex-antivortex pairs.

At low temperatures, only few vortices are present. These pairs can be treated as a weakly interacting gas (a dilute gas of logarithmically interacting vortex-antivortex pairs).

As temperature increases, so does the density of vortex pairs. This results in screening of the interaction between members of vortex-antivortex pairs with large separation. At some higher temperature this interaction is weakened enough so that these pairs start unbinding.

The presence of dissociated vortices destroys the quasi long range order of the correlation length and, apparently, its short range order can no longer support superfluid flow. At this vortex unbinding transition, the superfluid areal density vanishes. This temperature is usually ≤ 1K.

The essence of the Kosterlitz-Thouless theory can be explained by a simple energy-entropy argument [3].

Let us us consider a vortex excitation which consist of a quantised circulation flow field around a nonsuperfluid core. This flow satisfies the following condition:

$$\oint \vec{v_s} \cdot d\vec{l} = 2\pi n \frac{\hbar}{m} \tag{2}$$

where n is the vorticity.

For a single vortex, solving (2), one gets

$$v_s = \pm n \frac{\hbar}{m} \frac{1}{r} \tag{3}$$

The vortex energy consists of the kinetic energy of the flow field and of a temperature dependent energy associated with the core

$$\begin{aligned} E_1 &= \frac{1}{2} \int_{r_0}^{R} dr \rho_{s0} v_s^2 + E_c(T) \\ &= n^2 \pi K_0 \ln(\frac{R}{r_0}) + E_c(T) \end{aligned} \tag{4}$$

where R is the size of the system and $K_0 = \rho_{s0}(T)(\hbar/m)^2$.

The energy of a single vortex diverges as $\ln(R)$.

On the other hand, the energy of a vortex-antivortex pair is finite and is given by

$$E_{pair}(r) = 2\pi n^2 K_0 \ln(\frac{r}{r_0}) + 2E_c(T) \tag{5}$$

where r is the separation between the vortices (which is a finite number).

The entropy of a single vortex in a system of size R is given (approximately) by

$$S = k_B \ln\left(\frac{R^2}{r_0^2}\right) \tag{6}$$

as $\frac{R^2}{r_0^2}$ is the number of possible sites the core of the vortex can occupy.

Restricting our discussion to vortices with vorticity $n = 1$, the free energy of an isolated vortex is

$$F = E_1 - TS = (\pi K_0 - 2k_B T)\ln\left(\frac{R}{r_0}\right) \tag{7}$$

This equation suggests that at low temperatures the energy term dominates and hence single vortices are not favoured. Instead, they form pairs to minimize free energy. On the other hand, at high temperatures, where the entropy term dominates, the free energy for adding a single vortex (by unbinding a pair) is negative and hence single vortices are favoured.

So there must be a critical temperature at which the unbinding is energetically possible ($\Delta F_{\text{one unbinding}} \leq 0$).

$$T_c = \frac{\pi K_0}{2k_B} \tag{8}$$

For $T > T_c$, single vortices are created which, in turn, destroys correlation on long scales.

2 Static Kosterlitz-Thouless theory [4]

Discussed above was the basic idea of how the superfluid transition in helium films works. For better results one has to treat correctly the interactions of the vortex system and must include Bernoulli (or Magnus) forces on a vortex due to the velocity field of other vortices.

It is important to point out that this (static) theory is limited to systems without externally applied flow fields. Therefore oscillating substrate experiments, as well as, third sound experiments cannot be explained by this theory.

For vortices with opposite vorticity, this interaction is attractive and represents the binding energy of a vortex pair. This energy was given above by (5).

At sufficiently low temperatures we have a dilute gas of vortex pairs, and hence interactions between pairs can be ignored. In this case, the equilibrium density of vortex pairs with a separation r between pair members is determined by a Boltzmann distribution.

Assuming vortex cores do not overlap, the number of possible sites for the first vortex is $\frac{R^2}{r_0^2}$. If we fix the position of the first vortex, the second vortex must lie in an annular region a distance r from the first vortex. The number of possible sites is therefore $\frac{2\pi r dr}{r_0^2}$.

The areal density for vortex pairs with separation r is then given by

$$\begin{aligned} n(r) &= \frac{dN(r)}{R^2 dr} = \left(\frac{2\pi r}{r_0^4}\right) \mathrm{e}^{-\frac{E_{pair}}{k_B T}} \\ &= \frac{2\pi r}{r_0^4} y_0^2 \mathrm{e}^{-\frac{V(r)}{k_B T}} \end{aligned} \tag{9}$$

where $y_0 = \mathrm{e}^{-\frac{E_c(T)}{k_B T}}$ is the fugacity of the vortices and $V(r) = 2\pi K_0 \ln(r/r_0)$ is the interaciton potential for a vortex pair.

At higher temperatures the density of vortex pairs becomes larger and we modify (9) to include the interaction between different pairs. Screening effects must also be included, as the interaction between members of a pair with separation r is screened with intermediate pairs with smaller separation $(r' < r)$ situated between them.

As this behaviour of vortices is analogous to the behaviour of dipoles in a dielectric medium, we will apply the same terms and methods developed in the theory of dielectrics [4, 2].

Using the equilibrium Boltzmann distribution of pairs, one can define the polarizability of vortex pairs with separation r as

$$\alpha(r) = \frac{\pi r^2 K_0}{2 k_B T} \tag{10}$$

This expression is valid only in the weak field limit ($r\hbar\rho_{s0} v_s/m \ll k_B T$).

Given this polarizability, one then defines a length-dependent dielectric function $\epsilon(r)$ which modifies the unscreened force

$$F(r) = -\frac{dV(r)}{dr} \quad = \quad -\frac{2\pi K_0}{r} \quad \text{to}$$

$$F(r) = -\frac{2\pi K_0}{r\epsilon(r)} \tag{11}$$

Introducing susceptibility we have

$$\begin{aligned} \epsilon(r) &= 1 + 4\pi\chi(r) \\ \chi(r) &= \int_{r_0}^{r} dr' \alpha(r') n(r') \end{aligned} \tag{12}$$

where $\chi(r)$ is the susceptibility of all pairs with separation less than r. Note that in (12) we modified $n(r)$ to include the screened potential

$$n(r) = \frac{2\pi r}{r_0^4} y_0^2 \mathrm{e}^{-\frac{V'(r)}{k_B T}} \tag{13}$$

where

$$V'(r) = 2\pi K_0 \int_{r_0}^{r} dr' \frac{1}{r'\epsilon(r')} \tag{14}$$

Equations (12), (13) and (14) represent a self-consistent set of equations which allows one to evaluate the effective dielectric constant in the thermodynamic limit $r \to \infty$.

It is convinient to define the following length-dependent parameters

$$\begin{aligned} K^{-1}(l) &= \epsilon(r) k_B T K_0^{-1} \\ y(l) &= y_0 \left(\frac{r}{r_0}\right)^2 \mathrm{e}^{-\frac{V'(r)}{2k_B T}} \\ l &= \ln \frac{r}{r_0} \end{aligned} \tag{15}$$

Using these definitions one can rewrite equations (12 - 14) as follows

$$\begin{aligned} K^{-1}(l) &= k_B T K_0^{-1} + 4\pi^3 \int_0^l dl' y^2(l') \\ y(l) &= y_0 \mathrm{e}^{2l - \pi \int_0^l dl' K(l')} \end{aligned} \tag{16}$$

or in differential form

$$
\begin{aligned}
\frac{K^{-1}(l)}{dl} &= 4\pi^3 y^2(l) \\
\frac{y(l)}{dl} &= y(l)\left[2 - \pi K(l)\right]
\end{aligned}
\tag{17}
$$

with initial conditions

$$
\begin{aligned}
K(l)|_{l=0} &= \frac{K_0}{k_B T} \\
y(l)|_{l=0} &= y_0
\end{aligned}
\tag{18}
$$

These equations represent a recursion relation for the problem of 2D superfluidity.

$K(l)$ is the interaction between vortices separated by distance $r, (r \equiv r_0 e^l)$, taking into account the effects of screening by other pairs with separation less than r.

The quantity $K(l \to \infty)$ includes the effects of vortex pairs on all length scales and is proportional to the macroscopic superfluid density of the film.

$$
\begin{aligned}
K^{-1}(l \to \infty) = k_B T K_0^{-1} \epsilon(l \to \infty) \\
\rho_s(T) = \epsilon^{-1}(l \to \infty)\rho_{s0}(T)
\end{aligned}
\tag{19}
$$

The quantity $y(l)$ is related to the areal density of vortex pairs with separation r. Large $y(l)$ means large density of vortex pairs.

Solving the system of differential equations one obtains

$$
\frac{dK^{-1}(l)}{dy(l)} = \frac{4\pi^3 y(l)}{2 - \pi K(l)} \tag{20}
$$

After integration

$$
2\pi^2 y^2(l) - \frac{2}{\pi} K^{-1}(l) + \ln\left(\frac{K^{-1}(l)}{\pi}\right) = C(T) \tag{21}
$$

The intgration constant $C(T)$ is determined by the initial conditions at $l = 0$.

Using (18) one identifies this integration constant as

$$C(T) = 2\pi^2 y_0^2 - \frac{2}{\pi} k_B T K_0^{-1} + \ln\left(\frac{k_B T K_0^{-1}}{\pi}\right) \tag{22}$$

or in terms of the core energy $E_c(T)$ and background superfluid density $\rho_{s0}(T)$

$$C(T) = 2\pi^2 \mathrm{e}^{-\frac{2E_c(T)}{k_B T}} - 2\left(\frac{k_B T}{\pi\rho_{s0}(T)(\hbar/m)^2}\right) + \ln\left(\frac{k_B T}{\pi\rho_{s0}(T)(\hbar/m)^2}\right) \tag{23}$$

Substituting $y^2(l)$ in (17) from (21) one obtains the differential equation for $K^{-1}(l)$

$$\frac{dK^{-1}(l)}{dl} = 4K^{-1}(l) - 2\pi \ln\left(\frac{K^{-1}(l)}{\pi}\right) + 2\pi C(T) \tag{24}$$

This equation can be solved for $K^{-1}(l)$ numerically, while $y(l)$ is determined by (21).

On Fig.2 a family of trajectories in the (K^{-1}, y) plane is shown. Each trajectory corresponds to a different temperature with arrows indicating the direction of incresing l. The dotted line represent the locus of initial conditions corresponding to $l = 0$.

Let us analyze Fig.2.

We can distinguish three different types of trajectories.

For trajectories marked A, as we go from $l = 0$ to $l \to \infty$, $y(l)$ decreases and goes to zero. According to the definition of $y(l)$ this means that there are no vortex pairs with infinite separation, hence no vortices which are not bound to some other ones. This corresponds to the ordered state with $T < T_c$. These trajectories intersect the K^{-1} axis at some nonzero value, and therefore ρ_s remains nonzero although smaller than its initial value ρ_{s0}.

Trajectories marked C, are in sharp contrast with the previous ones.

As l increases, the trajectory initially flows towards smaller values of y, but eventually changes direction and flows towards increasing values of y and K^{-1}, never intersecting the K^{-1} axis. As $K^{-1}(l \to \infty)$ diverges, $\rho_s \to 0$.

The large values of y suggest that there is a high density of vortices and hence the system is in a disordered state. These trajectories correspond to temperatures $T > T_c$.

The single trajectory labeled B is the critical trajectory for which $T = T_c$. This is the highest temperature at which there still exists a finite, nonzero superfluid density. Since at higher temperatures , trajectories diverge, yielding a zero superfluid density, there should be discontinuity of ρ_s at $T = T_c$.

From (17) one can show that

$$\lim_{T \to T_c} K^{-1}(l \to \infty) = \frac{\pi}{2} \tag{25}$$

This, in turn, defines a universal constant

$$\lim_{T \to T_c} \frac{\rho_s(T)d}{T} = \frac{\rho_s(T_c^-)d}{T_c} = \frac{2m^2 k_B}{\pi \hbar^2} \tag{26}$$

where d is the film thickness and m the mass of the helium atom.

This is one of the principal results of the static Kosterlitz-Thouless theory. Equation (26) predicts a universal jump in superfluid density at the static transition temperature (see Fig. 1).

At this transition temperature only the vortex pairs with infinite separation can unbind. As the temperature increases, more and more pairs start unbinding.

3 Dynamic Kosterlitz-Thouless theory [4, 5]

In the previous section we assumed that the system of interacting vortices is in thermal equlibrium so external flow fields were not taken into consideration. However, the superfluid density is inherently the dynamic response of the system to an external flow field.

Third sound experiments involve both nonzero frequency and wave number excitations. Oscillating substrate experiments also have oscillating velocity fields with apparently nonzero frequency.

Therefore, in order to relate to experiments, one must modify the static Kosterlitz-Thouless theory and include effects of an external field on the motion of the vortices.

One simple way to solve this problem is to truncate the iteration of the recursion relations (16) at a scale l_ω which is the vortex diffusion length [5].

$$l_\omega = \frac{1}{2} \ln \left(\frac{14D}{\omega r_0^2} \right) \tag{27}$$

where ω is the angular frequency of the substrate oscillation and D is the vortex diffusivity. The temperature dependence of D is rather weak so we can assume a constant value of order $\hbar/m$.

Vortex pairs larger than this scale do not have time to equilibrate with the imposed velocity field and therefore do not participate in screening.

This results in a somewhat enhanced superfluid density, a broadening of the phase transition which is shifted to a slightly higher temperature , and a dissipation peak at the transition.

Until now, we have assumed that infinitesimal external fields are used to probe the vortex plasma. In practice, this is often violated, so we need to include nonlinear finite amplitude effects. It turns out that these finite amplitude effects give rise to a broadenind of the transition, an increase of the superfluidity dissipation in the vicinity of the transition and a shift of the transition to larger film thickness.

A more precise treatment of the dynamic response of the vortex system follows the previous prescription for the static theory but including a frequency dependence of all quantities.

A modified frequency dependent susceptibility and dielectric function has to be introduced:

$$
\begin{aligned}
\chi(r,\omega) &= \int_{r_0}^{r} dr'\alpha(r')n(r')g(r',\omega) \\
\epsilon(r,\omega) &= 1 + 4\pi\chi(r,\omega)
\end{aligned}
\tag{28}
$$

where $g(r,\omega)$ is the dynamic response function.

4 Experimental techniques [4, 6]

The most common experimental technique for investigating helium films is the torsional oscillator technique. It was first developed by Andronikashvili [7]. His experimental apparatus consisted of a stack of circular disks separated by small gaps. This was then placed into an aluminum cylinder and suspended from a long torsion fiber made of phosphor bronze. The period of oscillations included contributions from the moment of inertia of the heluim dragged along with the cell. By measuring changes in the period as a function of temperature, Andronikashvili was able to determine the temperature dependence of ρ_n.

Modern torsional oscillators are based on the same general principles. They are technically improved to get a higher natural frequency to facilitate signal detection and noise rejection(by using a stiff beryllium copper alloy for the rod), and much higher Q value to make it simpler to to stabilize the oscillator period. Figure 3 shows a schematic view of a typical experimental cell.

Helium films are adsorbed on a Mylar film substrate. A strip of the plastic Mylar film ($\sim 10\mu$m thick, $\sim$ 1cm wide and $\sim$ 10m long) is wound as a spiral on the axis of the torsional oscillator.

The oscillator is driven (nby electronic means) at its resonant frequency ($\sim$ 1 kHz). After a helium film of the desired thickness has been adsorbed on the Mylar substrate, a series of period and Q measurements are made to determine the temperature dependence of the superfluid mass and dissipation.

When the oscillator is cooled below the transition temperature the superfluid decouples from the torsion pendulum and the period of the oscillator decreases.

Figure 4 shows a typical experimental result. A change in period of torsional oscillations and the dissipation Q^{-1} in the neighbourhood of the transition clearly indicates a superfluid phase transition.

Another effective tool for probing the Kosterlitz-Thouless theory is the quartz-crystal microbalance method[8, 9]. This method consists of monitoring the frequency of the crystal oscillations, while continuously changing the thickness of the liquid helium film adsorbed on its surface[5].

Other experimental techniques involve measurements of the effective thermal conductance of a substrate coated with a helium film[10], third sound experiments[11] etc.

5 Summary

A brief review of both the static and the dynamic Kosterlitz-Thouless theory of transition in helium films was given. It has been shown that helium films have a nonzero superfluid phase transition temperature. At this temperature the superfluid density exhibits a jump. The main mechanism responsible for the disappearance of the superfluid phase is the unbinding of vortex-antivortex pairs.

The most important experimental techniques for probing the Kosterlitz-Thouless transition were also discussed.

6 Figure Captions

- **FIG. 1.** : Calculated superfluid density $\rho_s(T)$ for different normal densities. The dashed line was calculated assuming only vorex excitations. The solid curve represents the full dynamic theory[4].
- **FIG. 2.** : Flow trajectories derived from the Kosterlitz-Thouless recursion relations. Trajectories marked A, B, C correspond to $T < T_c$, T_c and $T > T_c$ respectively [4]
- **FIG. 3.** : Cross section of the torsional oscillator [4].
- **FIG. 4.** : The reduced period shift $2\Delta P/P$ and dissipation Q^{-1} near the superfluid transition temperature of 1.215 K [6].

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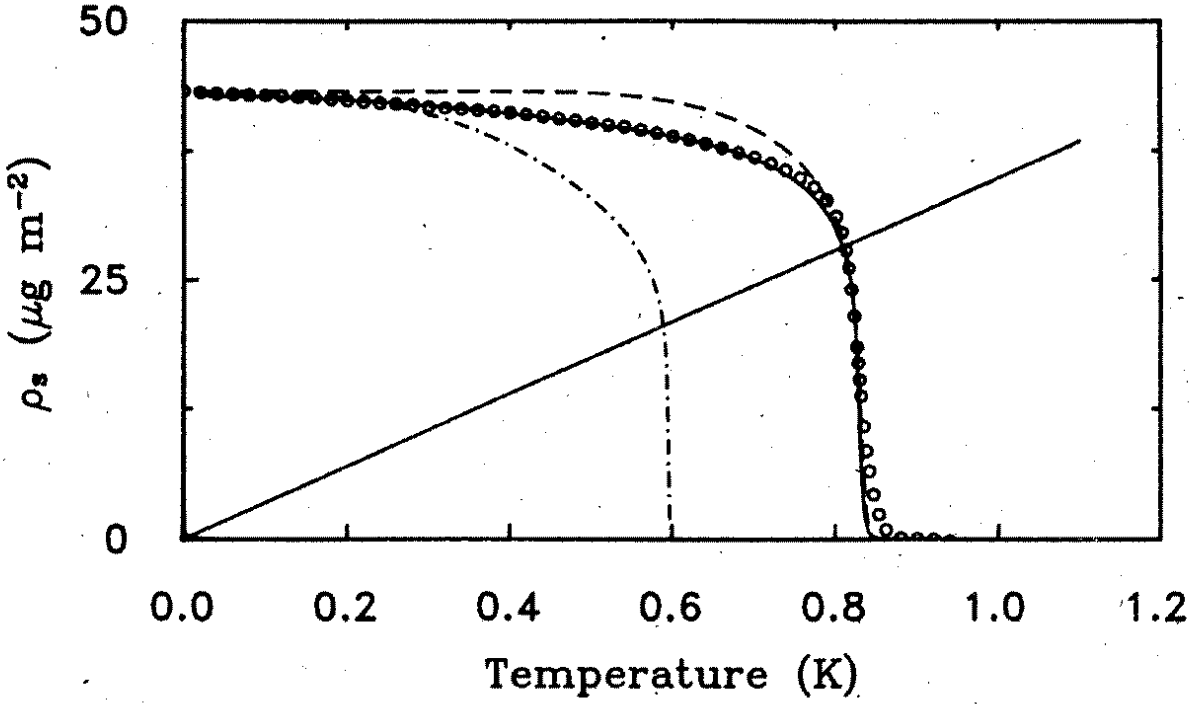
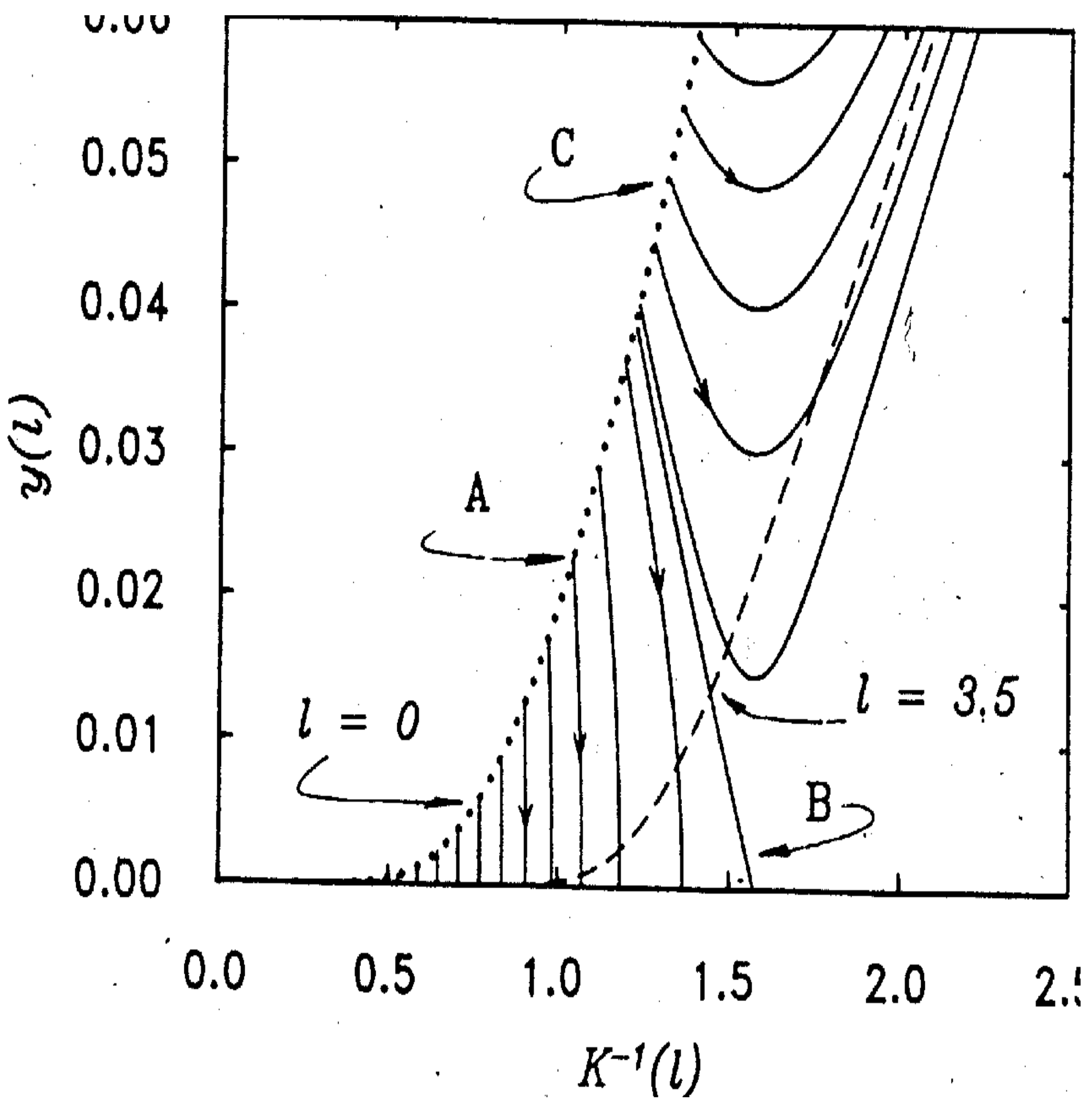


50
25
0
ρ_s (μg m^{-2})
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Temperature (K)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0.05
0.04
0.03
0.02
0.01
0.00
y(l)
0.0
0.5
1.0
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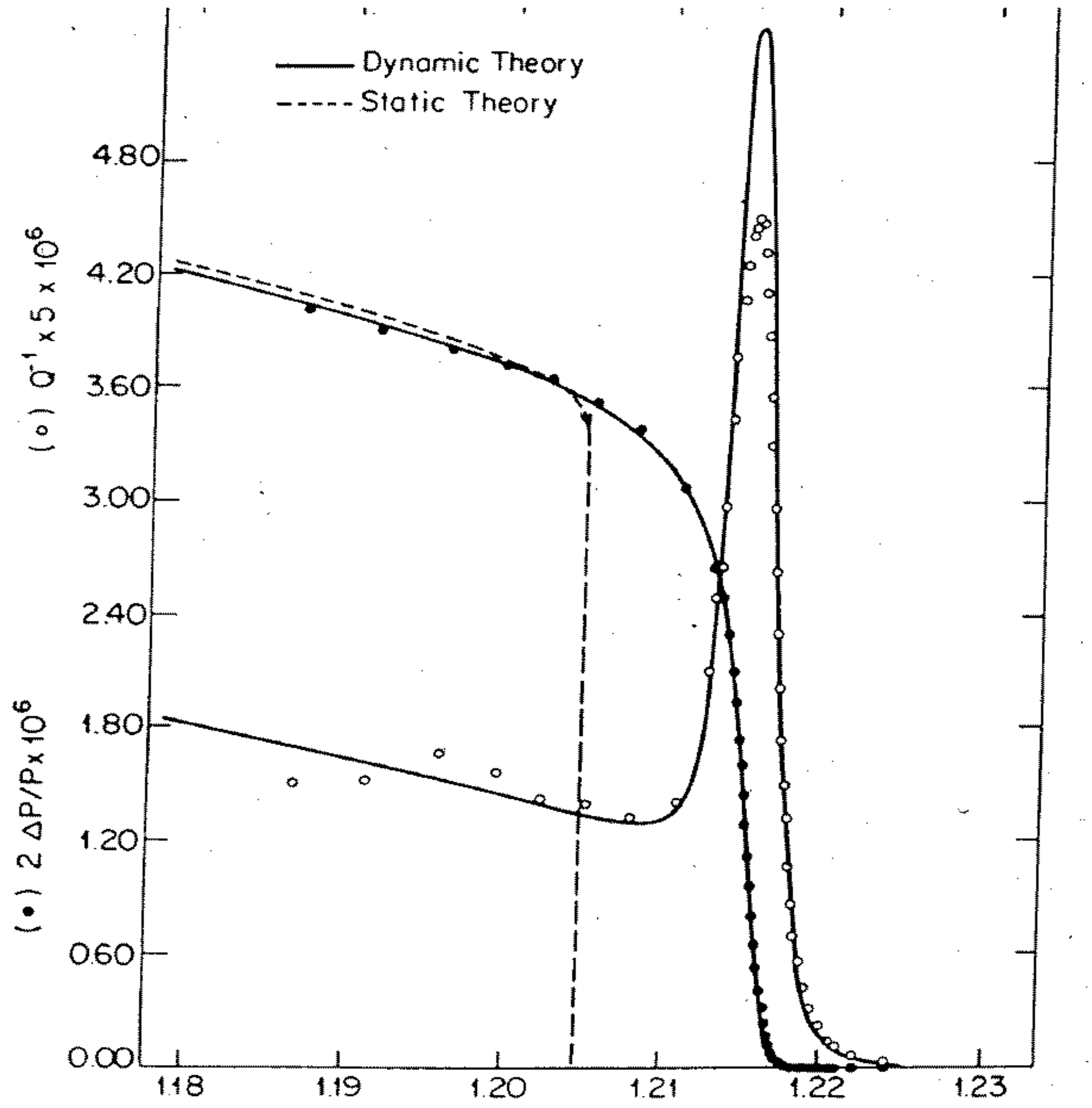
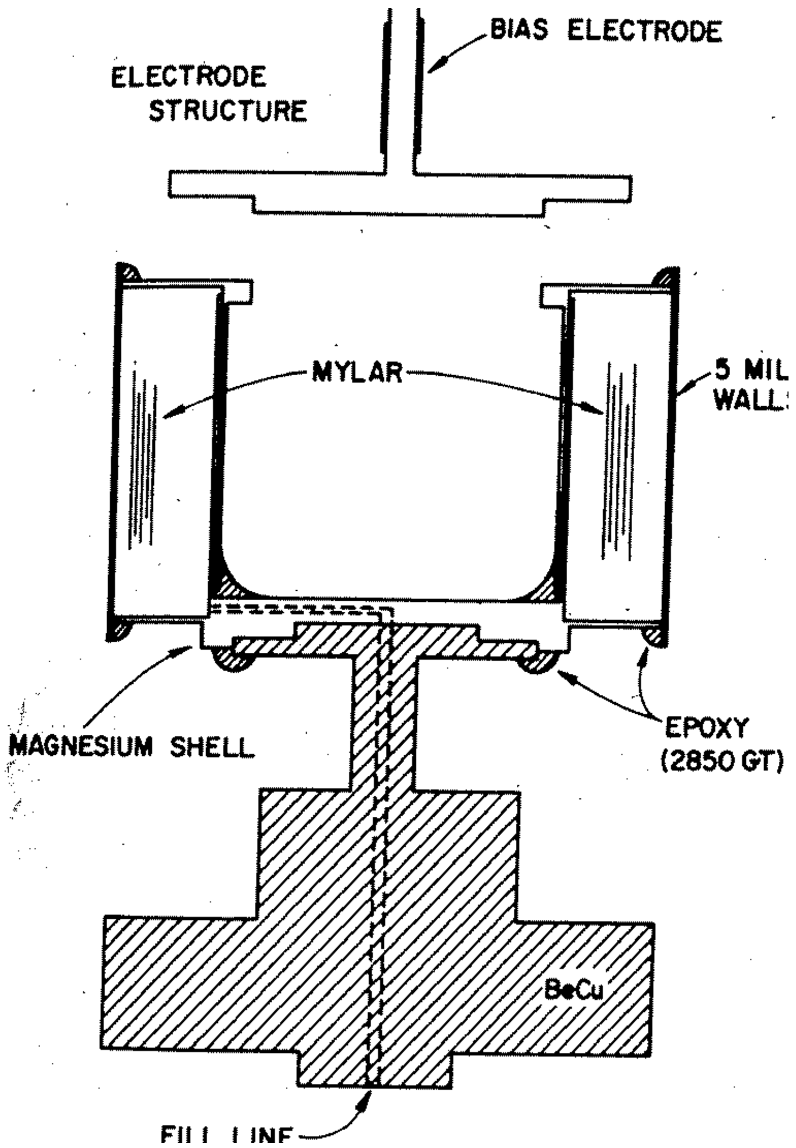
2.0
2.5
K^{-1}(l)
C
A
B
l = 0
l = 3.5



Dynamic Theory
Static Theory
4.80
4.20
3.60
3.00
2.40
1.80
1.20
0.60
0.00
(∘) $Q^{-1} \times 5 \times 10^6$
(•) $2\,\Delta P/P \times 10^6$
1.18
1.19
1.20
1.21
1.22
1.23



BIAS ELECTRODE
ELECTRODE
STRUCTURE
MYLAR
5 MIL
WALL
MAGNESIUM SHELL
EPOXY
(2850 GT)
BeCu