Phase Transitions

Homework Sheet 6

Due 10am Thur 3 April 2008, in the 563 box.

Question 6–1.

This question is a continuation of HW 4–1, in which the mean field theory for the Ising universality class is derived heuristically. Here, we use the mean field theory calculation of the Gibbs free energy to present the analogue of the Maxwell construction for magnetic systems, and to motivate the Landau free energy. The notation is given in HW4–1.

(a) Consider the case of uniform magnetisation $m_i = m$ on a d-dimensional hypercubic lattice, with coordination number $z = 2d$. Expand Γ to quartic order in m and show that there is a second order transition at $T_c = 2dJ/k_B$. From the equation of state, check the values of the critical exponents β and δ and verify they are what we expect in MFT.

(b) Sketch the form of $H(m)$ and $m(H)$ above and below the transition, *as given by the mean field theory.* Notice that your answer contains an unphysical portion below T_c. It is tempting to identify the Landau free energy with the function

$$L'(m, \hat{H}) \equiv \Gamma(m) - mN(\Omega)\hat{H}$$

where $\hat{H}$ is a parameter and not the function $H(m)$, and $m(H)$ is given by the mean field theory. Show that the condition that L' be minimised with respect to m implies the equation of state $\hat{H} = H(m)$. Sketch the form of $L'(m)$ above and below the transition for $\hat{H}$ positive, negative and zero. Hence show that the condition that L' be globally minimised removes the unphysical portion of the curve $m(H)$. This is Maxwell's equal area rule for magnetic systems. Notice the correspondence between the variables m and H in the magnetic case and p and V in the fluid case.

Question 6–2.

Here we will use our mean field solution of the Ising model to solve the lattice gas model of a fluid. Please refer to section 12.2.1 of my book to remind yourself the detailed basis for the correspondence between the Ising model and fluid models. The relevant section is attached to this PDF. The goal is to make a simple theory of the critical behaviour of a fluid. Set $U_1 = 0$, and check that you understand the correspondence between the lattice gas variables and the Ising variables. In particular, write down the relation between the pressure and the free energy of the Ising model. Also, write down the relation between the mean density ρ of the lattice gas and the mean value of the magnetisation of the Ising model.

(a) Express E_0 in terms of H and J. Using the result of 4–1, rewrite this in terms of H and T_c. Write down the relation between the pressure p, $H(m)$ and $S(m)$, using the results from 4–1 for the uniform magnetisation case. Hence show that the equation of state in the mean field approximation is

$$p = k_B T \log\left(\frac{1}{1-\rho}\right) - 2k_B T_c \rho^2$$

(b) Show that at the critical point for the fluid (p^*, ρ^*, T^*),

$$p^* = k_B T_c(\log 2 \; - 1/2),$$

and $T^* = T_c$, $\rho^* = 1/2$. This corresponds to the critical point $H = 0$, $T = T_c$ in the Ising model.