

Phase Transitions

Homework Sheet 6

Due 5pm Fri 28 April 2017, in the 563 box.

Question 6–1.

In this question, you are asked to perform a variant of the real space renormalisation group method – decimation – on the $d = 1$ Ising model, and to compare the results with exact results which you obtained from the transfer matrix method. We work with a system of N sites, N may be taken to be even, and we assume periodic boundary conditions $S_{N+1} = S_1$. The Hamiltonian is

$$\mathcal{H} = K \sum_{i=1}^N S_i S_{i+1} + h \sum_{i=1}^N S_i + NK_0.$$

The partition function is

$$Z_N\{\mathcal{H}\} = \text{Tr} e^{\mathcal{H}}.$$

- (a) The partition function can be expressed as a trace over all spins of a product of transfer matrices. Write down the transfer matrix for this case. Now perform the sum over the even numbered spins only, to obtain a new effective transfer matrix for a system with twice the lattice spacing as the original system. Hence calculate the renormalised coupling constants K' , h' and K'_0 in terms of the original coupling constants. Verify that your expression for the renormalisation group transformation preserves the symmetries of the original problem. Why did we need K_0 ?
- (b) Set the field $h = 0$. Show that the recursion relation for K is

$$e^{4K'} = \cosh^2(2K)$$

and find the fixed points and sketch the flow. Work with the variable $w \equiv e^{-2K}$.

- (c) Linearise the recursion relation and find y_t .
- (d) Now set $h \neq 0$ and find the fixed points, the flow diagram and the exponent y_h .
- (e) What do your results mean for the behaviour of the $d = 1$ Ising model? Use the exact results from the transfer matrix calculations done in class to calculate y_h and y_t .

Question 6–2.

This question concerns the use of finite size scaling to estimate critical exponents and transition temperatures from transfer matrix calculations in a strip. Consider the $d = 2$ Ising model on a square lattice. There are N rows parallel to the x axis and M rows parallel to the y axis. We will require that $N \rightarrow \infty$ whilst we will calculate the transfer matrix for $M = 1$ and $M = 2$. Periodic boundary conditions apply in both directions, so that our system has the topology of a torus. The Hamiltonian is

$$\mathcal{H} = K \sum_{n=1}^N \sum_{m=1}^M S_{mn} S_{m+1n} + S_{mn} S_{mn+1}.$$

In a previous exercise, you constructed the transfer matrix for this problem, and calculated the eigenvalues λ_1 and λ_2 . The correlation length is given by

$$\xi^{-1} = \log \lambda_1 / \lambda_2.$$

Use finite size scaling and the results from HW 3–2 to estimate the critical value of K and the exponent ν .

Question 6–3 OPTIONAL - NOT GRADED.

This question requires you to investigate the fact that there is a shift (i.e. ‘renormalization’) in the transition temperature when Gaussian fluctuations are included. The starting point is again the derivation of mean field theory from the Hubbard-Stratonovich transformation. There we showed that the partition function of an Ising system could be written as the functional integral

$$Z = \int_{-\infty}^{+\infty} \prod_{i=1}^N d\psi_i e^{-\beta S(\{\psi_j\}, \{H_j\}, \{J_{ij}\})}.$$

In mean field theory, the integral was replaced by the integrand evaluated at the extremum $\bar{\psi}_i$.

- (a) Expand the functional S to second order in $\psi_i - \bar{\psi}_i$ and evaluate the resultant Gaussian integral for the partition function. Hence show that the lowest order correction (first term in an expansion sometimes known as the loop expansion) to mean field theory gives for the free energy F

$$F = S(\{\bar{\psi}_i\}) + \frac{1}{2\beta} \log \det [\delta_{ij} - \beta (1 - \tanh^2(\beta \bar{\psi}_i) J_{ij})],$$

where irrelevant constants have been dropped from the partition function.

- (b) Calculate the Gibbs free energy in this approximation. You will find that the algebra is eased by writing the second term above as $\epsilon(\delta\Gamma)$, where ϵ is a dummy variable introduced to keep track of orders of approximation, and working to $O(\epsilon)$ only. Also, there is no need to explicitly evaluate $\partial(\delta\Gamma)/\partial H_i$. It actually drops out of the calculation!
- (c) In mean field theory, the susceptibility $\chi_T \propto t^{-1}$. From your answer to (b), evaluate χ_T^{-1} (formally) and hence show that it vanishes at a *negative* value of t . This is the shift in the transition temperature due to Gaussian fluctuations. Please give a physical reason why it makes sense that the T_c has shifted in the direction that you found.