Phase Transitions

Homework Sheet 1

Due 5pm, Fri 3 Feb 2017 in the 563 box.

Please attempt these problems without referring to textbooks, although you may use your notes. The most efficient way to learn is to attempt a question and then if you are stuck, read the relevant section of the notes, then close the notes and try again. I would not recommend starting this homework the night before it is due. Please do not refer to solutions of these problems provided by earlier students in this class.

Question 1–1.

(a) By noting that the area of a right-angled triangle can be expressed in terms of the hypotenuse and (e.g.) the smaller of the acute angles, prove Pythagoras' theorem using dimensional analysis. You will find it helpful to construct a well-chosen line in the right-angled triangle. *Note: the whole point of dimensional analysis is that you do NOT need to solve for the functional form of the solution to a given problem. Thus, in this question, you must pretend that you do not know trigonometry.*

(b) Now consider the case of Riemannian or Lobachevskian geometry (i.e. the triangle is drawn on a curved surface such as a riding saddle or a football). Does your solution to (a) still work? If not, why?

Question 1–2.

This exercise walks you through the estimation of some complex and interesting scaling law problems in fluid dynamics. It has several parts and be sure to answer all the sub-questions asked in them please.

(a) The viscosity η of a classical plasma of singly-charged ions of mass m_i and electrons might be expected *a priori* to depend upon temperature T, m_i, the density of ions n and the electronic charge e. In fact, it is found that to a good approximation, η is independent of n. Using this finding, show that

$$\eta \propto \frac{m_i^{1/2}(k_B T)^{\alpha}}{e^4}.$$

and determine the value of the exponent α.

(b) Now consider neutral systems at non-zero temperature T, where n is the number of particles per unit volume. First, let's work in the classical limit, where the interaction of particles is described by the scattering cross section σ. Show that dimensionally, shear viscosity η has units of momentum/area. Estimating the momentum of a classical gas, using de Broglie's relation, show that the shear viscosity of a classical gas is independent of n and $\hbar$, and scales as $T^{1/2}$.

(c) Now consider a quantum Fermi gas. In atomic traps, Fermi gases can be tuned to be in the so-called unitary limit, where they are very strongly interacting and the s-wave scattering length is divergent. Assuming Type I asymptotics, the s-wave scattering length does not play a role in the problem any more. There are then only two length scales that can come into the problem. What are they? Hence show that at temperatures much smaller than the Fermi temperature, $\eta \propto n$ and thus independent of T. Then show that at temperatures much larger than the Fermi temperature, $\eta \propto T^{\gamma}/\hbar^2$, and determine the exponent γ.

Question 1–3.

It is 1947 and you are a spy for superpower R. You notice in *Life* magazine a series of time lapse photographs of the early stages of the first test of an atomic bomb, at Trinity, New Mexico. They are reproduced at:

http://guava.physics.uiuc.edu/~nigel/courses/563/Trinity

The photographs show the expansion of the shock wave caused by the blast at successive times in ms. Assuming that the motion of the shock is unaffected by the presence of the ground, and that the motion is determined only by the energy released in the blast E and the density of the undisturbed air into which the shock is propagating, ρ, derive a scaling law for the radius of the fireball as a function of time. Extract data from the photographs (do it yourself – do not just copy what is written in my book!) to test your scaling law and hence deduce the yield of the blast. **You must test your scaling law by plotting a graph. You should consider carefully and then explain what is the most useful graph to plot.** You should assume that all numerical factors are of order unity. This information will not be declassified for another 3 years, so you may reasonably expect promotion and other rewards for your efforts.

Question 1–4.

How fast does a river flow? In North America, hydraulic engineers have found that they can make satisfactory predictions using the empirically-obtained formula (due to Manning)

$$V = \frac{1.486}{n} R^{2/3} S^{1/2}$$

where V is the mean flow velocity in feet/sec, S is the slope of the river (expressed as S feet/feet, meaning that for every foot travelled horizontally, the river rises or falls by S feet), and R is the hydraulic radius in feet, defined as the cross-section area of the river, divided by the perimeter that is in contact with the water. If the river has a rectangular cross-section, of depth h and width w, then R is simply $hw/(w+2h)$. Manning's coefficient n accounts for the roughness of the river bank and bottom, and is tabulated for a variety of environments through which the river flows. For example, some typical values reported in the literature are $n = 0.015$ (brickwork), $n = 0.04$ for a gravel bed stream. Physically, these different environments are not geometically smooth walls, but have some pattern of roughness on them, which we will assume has a scale r. Note the eccentric use of English units in the Manning formula. In this question, we will try to understand the systematics of the engineers' empirical formula, so that, for example, we can work out how quickly rivers flow on Mars.

(a) Does the factor $1.486/n$ have units?

(b) Rivers flow under the action of gravity, so that the effective gravitational acceleration experienced is $g \sin\theta \approx gS$ where θ is the angle the river channel makes with the horizontal. We expect that $V = F(R, g, S, r)$. Making sensible, physically-motivated choices for dimensionless variables, determine the form of V.

(c) In (b) you should have introduced a new function of a single variable, that we will call $f(z)$. What must be the asymptotics for small or large z (depending on how you did the dimensional analysis) in order for it to reproduce the empirical Manning formula?

(d) Hence calculate how Manning's coefficient depends on the physical roughness r and g. This information would be useful for future landscape engineers on Mars ...