

Phase Transitions in Granular Material

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Abstract

Granular fluids and solids are different from conventional fluid and solids in many ways. The granular solid-fluid transition has been induced in two ways. First, the granular solid is tilted until an avalanche forms. This appears to be a first order transition. Second, granular solids can be shaken. This appears to be a second order transition. The liquid-gas transition has also been studied. This transition involves the inelastic collapse of particles. Various order parameters have been proposed for both types of transitions. Finally, a Landau free energy has been written down for a granular system, and various critical exponents calculated.

1 Introduction and motivation

Granular materials are composed of large number of macroscopic particles that interact through short range, repulsive, interactions. Quantum mechanical effects do not appear. The principle interaction between different grains is that they cannot occupy the same space at the same time.

Granular materials are a physically interesting system to study. Traditional thermodynamics do not apply to granular materials. In ordinary gases at room temperature, the relevant energy scale is $k_B T$. However, because grains are much more massive than gas molecules, the potential energy due to gravity is more important to them than $k_B T$. The energy scale for granular materials is mgd where d is the diameter of a grain. For a grain of sand at room temperature mgd is 10^{12} the size of $k_B T$.



Figure 1: This is a solid-fluid phase transition in a grain elevator. This particular instance of phase transitions is actually fairly common and destructive, and of particular concern to Midwesterners.

Granular materials are a useful system to study. They are widely used in industry; a major source of industrial waste is the inefficient transportation of granular materials. For example, paint factories use pipes to transport powders. The “fluid” granular material occasionally changes phase to a “solid” granular material, causing the pipe to jam. Factories do not know how to keep the pipes from jamming, and they usually unjam them by banging on the pipe with a wrench.

Granular materials are quite common in Illinois because wheat and other food grains are granular materials (see fig. 1).

In the natural world, an example of a granular phase transition is an avalanche. Predictive powers about avalanches would be useful because of the damage they sometimes cause.

2 Characterization of solid and fluid phases

The phases of granular materials are defined by the interactions of the grains.

Granular solids are grains which are in contact but not moving. If the grains were not in contact they would not be a granular material because the grains would not be interacting. There appears to be one solid phase, but the density of a granular solid can show a great deal of variation.

Granular fluids interact through space filling and inelastic collisions. Qualitatively, they are collections of grains hit each other and loose energy in each collision.

3 Why is there new physics here?

Given the many similarities of granular materials to conventional fluids and solids, one may ask why we need new physics to describe granular materials.

Granular materials lose energy much more quickly than molecules. Molecules collide almost elastically, while grains collide inelastically. Once a normal fluid begins to flow, it will do so for quite some time. Granular fluids, on the other hand, leak energy. They must be continually fed with energy to keep moving. Therefore, granular materials do not vary slowly. Slow variation is necessary to derive hydrodynamic equations.

In experiments where granular materials are shaken, there is rapid variation at the boundary of the granular material during the shakes. Again, this rapid variation prohibits the derivation of hydrodynamic equations.

There has been some success in developing a hydrodynamics for a 2 dimensional granular gas. However, the hydrodynamics worked only when the grains were gas-like, e. g. when the collisions were almost elastic.

In equilibrium statistical mechanics, we said that the initial state of the system did not affect the system. The past states of the granular system *do* affect the current states of the system. (Granular materials are not unique in this respect. Glasses also exhibit history dependence.) A series of experiments were conducted in which a long tube was filled with granular material. The tube was then shaken with different maximum accelerations. Between the shakes, the grains did not move. However, the maximum density of the granular material (measured by measuring the capacitance) was determined by the acceleration of the shake. In other words, the state of the system was determined by how the system was prepared. This indicates a history dependence in the system.

For these reasons, the physics of phase transitions in granular materials is different from that of phase transitions in conventional fluids and solids.

4 Solid-fluid transition

4.1 Avalanches

Avalanches are an example of a solid-liquid phase transition. In avalanches, the entire system does not undergo the phase transition, only the uppermost layers of the granular material “melt”.

Avalanches have been experimentally produced by filling a half-tube with grains and then rotating it around its axis (see fig. 2). This causes the angle of the “pile” to steadily increase.

It has been found that there are two critical angles. The angle of repose, θ_r , which

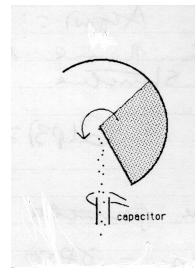


Figure 2: Experimental apparatus used to study avalanches

is the angle sandpiles achieve after an avalanche, and the maximum angle of stability, θ_m . Experimentally, avalanches occur at reasonably constant time intervals, which indicates that θ_r and θ_m are well defined (see fig. 3).

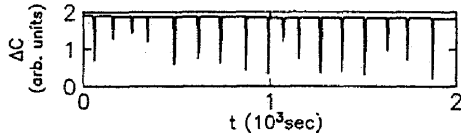


Figure 3: The depth of the spike indicates the amount of material that fell off the granular material. The horizontal axis is time.

An avalanche is a first order transition. There are two states (angles) which are local minimas of the free energy. Given some perturbation, the system will jump the free energy barrier between the states.

In fact, one can use Newtonian mechanics to describe the system Applying Newtons second law to a single grain we get

$$ma = mg \sin \theta - F \quad (1)$$

where $mg \sin \theta$ is proportional to the driving force and F is the retarding force on a bead. Estimates of F have been made. $F = \frac{dE}{dx}$, the amount of energy a bead loses per distance it travels along the surface of the granular material. (x is in the direction of the slope of the granular material.)

Energy has two parts, kinetic and potential. $F_{\text{kinetic}} = \frac{\Delta E}{\Delta x} = \frac{\frac{1}{2}m(\dot{\gamma}d)^2}{\lambda}$ where $\dot{\gamma}d$ is the relative velocity of two layers a distance d apart and λ is the average distance a bead travels before it looses $\frac{1}{2}m(\dot{\gamma}d)^2$ of energy to collisions.

This can be restated in terms of dimensionless variables: $\dot{\gamma} = \dot{\gamma} \sqrt{\frac{d}{g_{\perp}}}$ and $\beta = \frac{d}{2\lambda}$.

($g_{\perp} = mg \cos \theta$.) We also scale F_k ; $\tilde{F}_k = \frac{F_k}{mg_{\perp}}$. Now $\tilde{F}_k = \beta \dot{\gamma}^2$.

It is more difficult to calculate the potential term of a grain. First we estimate it when there is a high relative velocity between layers. The layer of grains a grain travels over is not smooth. One can imagine it like a sequence of hills and valleys. If the grain is moving fast enough with respect to the layer below it, it will not fall all the way to the bottom of the valley, instead, it will fly through the air to the opposite incline. We get that $\Delta h \sim \frac{g_{\perp}}{2\dot{\gamma}^2}$. So, at high relative velocities, $F_{\text{potential}} = \frac{mg_{\perp} \Delta h}{d} \sim \frac{mg_{\perp}}{2\dot{\gamma}}$.

When a grain layers have small relative velocity F_p tends to a constant, because the grain does not jump at all, but instead goes up and down each hill and valley. $F_p \sim \alpha mg_{\perp}$. The high and low relative velocity results are combined in a scaled force $\tilde{F}_p = F/(mg_{\perp}) \sim \frac{\alpha}{1+\alpha\dot{\gamma}^2}$.

Combining the kinetic and potential terms $\tilde{F} = \beta \dot{\gamma}^2 + \frac{\alpha}{1+\alpha\dot{\gamma}^2}$. We find a value of $\dot{\gamma}$ that minimizes \tilde{F} , call it $\dot{\gamma}_o$. $\dot{\gamma}_o$ is non-zero if and only if $\alpha\alpha > \beta$. If

we have a nonzero relative velocity that minimizes \tilde{F} then there are four steady-state solutions to our original Newtonian force equation (eqn. 1). If the driving force is small, the acceleration of the particles is 0. For stronger driving forces, there are three solutions; one again at $\dot{\gamma} = 0$. Another is in the region where $\frac{dF}{d\dot{\gamma}_o} < 0$, the third is in the region where $\frac{dF}{d\dot{\gamma}_o} > 0$. The solution with $\dot{\gamma}_o$ in the region of negative slope is unstable. The two remaining solutions, at $\dot{\gamma}_o = 0$ and $\dot{\gamma}_o > 0$ account for the hysteresis of the system (e.g., the difference between θ_m and θ_r).

A caveat to these experiments is that the behavior breaks down for sufficiently small systems because the angle of the sandpile is no longer clearly defined. The pile can no longer tell the difference between θ_r and θ_m .

4.2 Horizontal shaking experiments

A three dimensional granular material was prepared and horizontally shaken. Specifically, a shaker giving a sinusoidal vibration was filled with around 700 glass spheres, each with a diameter between 0.52 and 0.6 mm. There were 3-5 layers of spheres in the shaker.

The quantity measured is the pixel change. Pictures are taken at a rate of 25 Hz. Each pixel is given a value of 1 if it is a grain and 0 if it is background. Consecutive pictures are then subtracted, and the mean square of each point is recorded; it is called the pixel change. A pixel change of 0 would mean that the grains were not moving.

Grains were shaken with an amplitude of 20 mm, (3 and 4 grain diameters). Below a frequency of 2.9 Hz, the pixel change is almost a constant 0. Above 2.9 Hz, the pixel change is positive and looks linear. Pixel change is a possible order parameter for the system.

Hysteresis would indicate a first order phase transition because it would suggest that there are two local minimas of the free energy and that there is a region in phase space where the system is stable in either minima. To investigate possible hysteresis, the pixel change was measured at frequencies which was obtained through both increasing and decreasing the frequencies (see fig. 4). There is no hysteresis evident on this graph, therefore this is probably a second order phase transition.

4.3 Horizontal shaking simulations

A computer simulation similar to the experiments described in the last subsection was done. The normal force is proportional the the amount the grains have been deformed by pushing them together and to their speed towards each other. Mathematically, the normal force between two particles, i and j , was taken to be

$$F_{ij}^n = -k_n(d_i + d_j - \vec{r}_{ij}\hat{n}) - \gamma_n\vec{r}_{ij}\hat{n} \quad (2)$$

where d_i and d_j are the radii of the particles i and j , r_{ij} joins their centers, and \vec{r}_{ij} give their relative velocities. \hat{n} is the normal unit vector. k_n is related to

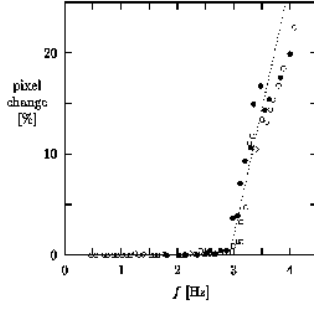


Figure 4: Frequency versus pixel change. The filled circles were approached from higher frequencies and the open circles were approached from lower frequencies.

the materials stiffness in the normal direction. γ_n is related to the amount of energy lost in the normal direction in a collision, experimentally, this would be related to the coefficient of restitution.

The shear force is in the direction of the grains relative velocities. It is proportional to the lesser of the frictional force between the grains and the energy lost in the collision. Mathematically, the shear force between two particles i and j is taken to be

$$F_{ij}^s = \text{sgn}(\vec{v}_{ij} \hat{s}) \min(\gamma_s \vec{v}_{ij} \hat{s}, \mu |F_{ij}^n|) \quad (3)$$

γ_s is related to the amount of energy lost in the shear direction in a collision. μ is the static friction in the shear direction. \hat{s} is the unit shear vector.

In simulation, $\gamma_s = 1Ns/m$; this is so large it made all shear forces frictional forces between the beads. $k_n = 10^5$. Other parameters of the system were set using experimental values. γ_n was taken to correspond to a restitution coefficient of 0.75. The coefficient of friction was taken to be $\mu = 0.1$ between particles and $\mu = 0.13$ between particles and walls. The container was taken to be 10.08 cm wide with period boundary conditions in the direction of shaking. The box was filled with 343 grains in two layers, with diameters ranging from 0.52 mm to 0.6 mm.

The results of this simulation are described in the next section.

4.4 Proposed order parameters

4.4.1 Granular Temperature T_g

In the simulation, the velocity fluctuations were measured. The average velocity was defined as

$$\langle \vec{v} \rangle = \frac{1}{N} \sum_{i=1}^N \vec{v}_i \quad (4)$$

The velocity fluctuation was defined as

$$T_g = \frac{1}{N} \sum_{i=1}^N \frac{m_i}{2} (\vec{v}_i - \langle \vec{v} \rangle)^2 \quad (5)$$

T_g is averaged over several cycles. T_g is sometimes referred to as the *granular temperature*. (Note that velocity is being measured in the reference frame of the container.)

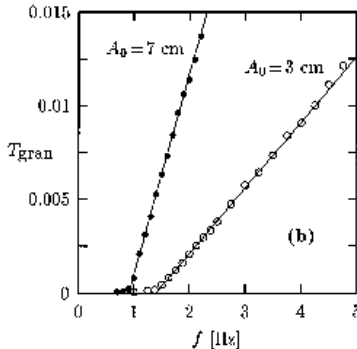


Figure 5: Frequency versus granular temperature

Again, the granular material is a solid up to a critical frequency (see fig. 5). The simulated critical frequency (1.87 ± 0.3 Hz) is less than the experimental critical frequency. However, both show the same qualitative behavior. A granular solid up to a certain critical frequency, and then a second order transition to a granular fluid.

The critical frequency increases with increases μ and decreases with decreasing μ . However, changing k_n had no effect on the critical frequency.

The granular temperature and pixel change are clearly related. Pixel change measures how much the grains have moved. Granular temperature measures velocity fluctuations near the transition point, this measures the fluctuations from 0 velocity (unless the entire granular material begins moving together, which seems unlikely.)

4.4.2 Pair distribution function, $G(r)$

The pair distribution function is defined as

$$G(r) = \frac{V}{N^2} \left\langle \sum_{i=1}^N \sum_{j=1, j \neq i}^N \delta(|\vec{r} - \vec{r}_{ij}|) \right\rangle \quad (6)$$

This function counts the number of pairs of particles which are separated by a distance \vec{r} . If one were plotting $G(r)$ for a granular solid, one would see double

peaks around integer multiples of the grain diameter. This is because the grains have formed a two-dimensional hexagonal crystal. The peaks of $G(r)$ decrease with increasing n if the grains are not perfectly crystallized. In an imperfect crystal, at larger \vec{r} , noise drowns out the crystal's signal.

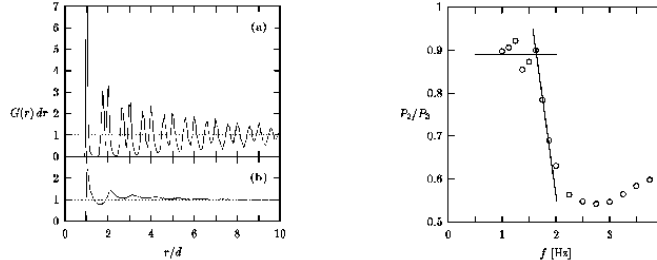


Figure 6: Left: Pair distribution function with an amplitude of 2 cm at (a) low and (b) high frequency. Right: The ratio of the 2nd and 3rd peak of the pair distribution function with an amplitude of 2 cm.

One can plot the ratio of the height of the second peak of $G(r)$ at $r = \sqrt{3}d$ (P_2) to the height of the third peak at $r = 2d$ (P_3). This was done for a shaking amplitude of 20 mm and plotted as a function of frequency. Again we see a phase change. In an infinite crystal, the ratio of the peaks is 1. However, P_2 decreases more quickly as the fluid phase takes over. There is a sharp drop in P_2/P_3 between 1.6 and 2 Hz (see fig. 6). One may consider this drop-off to be associated with the phase change. This phase change occurs at a different critical frequency than was identified by either the pixel change or the granular temperature.

This order parameter is of limited use because it measures the amount of crystallization. A granular solid need not be a crystal. Instead, it can be a random pack, which would not show the characteristic $G(r)$ described. Experimentally, if one prepares a granular solid with only one or two layers, there will be a great deal of crystallization. However, granular solids which occur naturally or in industry are seldom so small.

5 Liquid-gas phase transition

5.1 Inelastic Collapse

Let's remember some introductory physics. When a ball is dropped, the ball makes an inelastic collision with the group and bounces back up. The speed immediately after the n th bounce is $s_n = \eta s_{n-1}$ where η is the coefficient of restitution. By iteration, we can see that the speed of the ball after the m th bounce is $s_m = \eta^m s_0$. Furthermore, the height of the ball after the m th bounce (from energy conservation) is $h_m = s_m^2 / (2g) \sim \eta^{2m}$. Since $x = s \Delta t_m + \frac{1}{2} g (\Delta t_m)^2$

we get that $\Delta t_m \sim \eta^m$. Now we can add up the amount of time it takes for an infinite number of bounces. $T = \sum_{m=1}^{\infty} \Delta t_m \sim \sum_{m=1}^{\infty} \eta^m$. Since $\eta < 1$ this is a convergent series. We get an infinite number of collisions in a finite amount of time! This is known as inelastic collapse.

The same phenomena, an infinite number of collisions in a finite amount of time, is seen in granular material when the coefficient of restitution is sufficiently small. This has been observed principally in 2 dimensional, continuum, computer simulations (see fig 7).

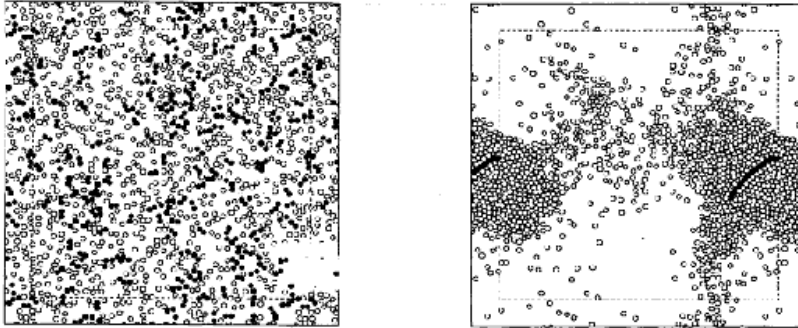


Figure 7: On the left $\eta = 0.99$, and there is no collapse. On the right, $\eta = 0.6$, and inelastic collapse is visible. Grains that collided in the last 200 collisions of this simulation are colored black.

Qualitatively, granular materials experiencing inelastic collapse look like two phases coexisting. Essentially, when the coefficient of restitution is sufficiently small, the grains lose energy very quickly, the clumped regions are regions where the grains have lost most of their energy.

5.2 Analytic Results

Some analytic work has also been done on this phenomena in granular materials. Zhou et. al. wrote down some standard kinematic equations for inelastic collisions between three particles, assuming that the central particle collides with first one, than the other, of its neighbors. Then he begins to look for fixed points in the velocity components in the line of the centers of the two colliders. The presumption is that the fixed points correspond to inelastic collapse. He assumes that the time between collisions goes to 0. After some algebra, one gets that

$$v_{n+1} = kv_n \quad (7)$$

(where v is the component of velocity in the direction of the line of centers) and that k has two possible values

$$k_{\pm} = \frac{1+\eta}{4} \cos \theta \pm \sqrt{\left(\frac{1+\eta}{4} \cos \theta\right)^2 - \eta} \quad (8)$$

where θ is the angle between the two center lines (see fig. 5.2).

For physical reasons, k must be a real number, therefore, $(\frac{1+\eta}{4} \cos \theta)^2 - \eta$ must be positive. With this criterion, one can disallow inelastic collapse for certain values of η and θ . Additional analytic work can be done to discern three different possibilities: inelastic collapse allowed, inelastic collapse not allowed, and “collapse” where the particles separate before having undergone an infinite number of collisions (see fig. 9).

5.3 Continuum Simulations in 2D

Continuum computer simulations have been made of the transition between the state with many small clusters to the state with one big cluster as the energy of the system decreases. Clustered particles are not necessarily inelastically collapsed, although (as can be seen in figure 9) clusters often form around inelastically collapsing particles.

In one 2-dimensional simulation, disks were confined between two hard walls 30 bead diameters apart which are at a fixed temperature. When a grain hit a wall, it was shot out. The velocity components were chosen from the distribution $P_{\perp} = \frac{1}{\sqrt{2\pi}} e^{-v_{\perp}^2/2T}$ and $P_{\parallel} = v_{\parallel} e^{-v_{\parallel}^2/2T}$ where T is the temperature of the wall. The simulation included 500 particles, and reached a steady state after around 10^5 collisions. The temperature, T , is defined as the mean kinetic energy per grain.

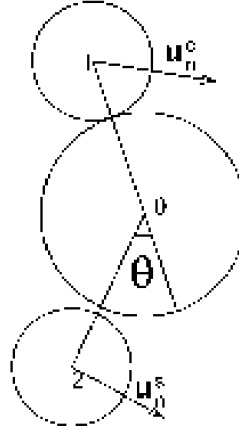


Figure 8: This picture defines θ .

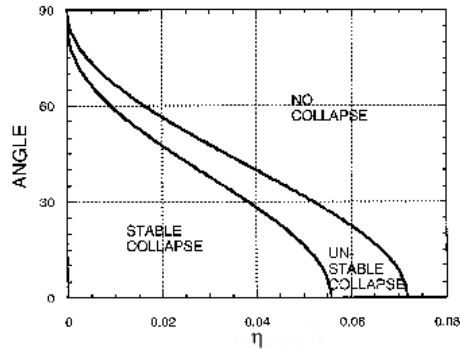


Figure 9: Where inelastic collapse is allowed and stable.

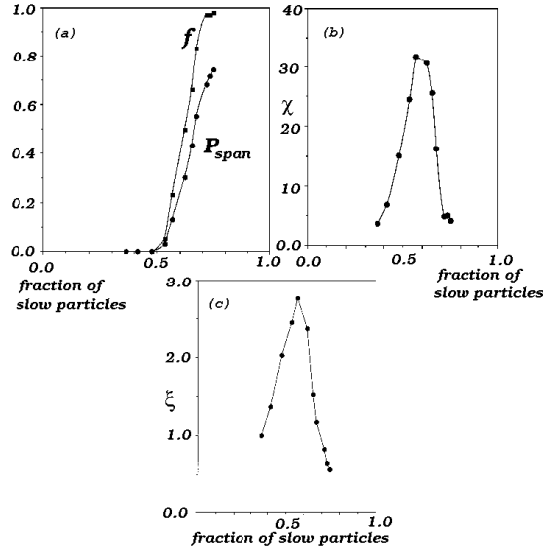


Figure 10: **a:** f is the fraction of runs which reach a spanning cluster. P_{span} is the fraction of particles contained in that cluster. **b:** χ is the mean non-spanning cluster size. **c:** ξ is the average diameter of non-spanning clusters, measured in disk diameters. The statistical errors in ξ and χ are around 10%.

A cluster is defined by a group of particles all with kinetic energy less than $0.01 T$. Two particles are part of the same cluster if their centers are less than 1.05 radii apart.

The simulation was run many times. Various measurements were made of the steady state (see fig. 10). ξ is to be the diameter of non-spanning clusters. (Note that this is a different definition than was given in equation 11.) χ is the average size of the non-spanning clusters. P_{span} is the fraction of occupied space which is taken up by the spanning cluster. These definitions make sense. χ measures how likely it is that a grain will be in a cluster. Since all the particles in a cluster are correlated, the defined ξ is a measure of the correlation length.

The graphs (fig 10) look like a pre-thermodynamic limit “phase transition” That is, the susceptibility χ and the correlation length ξ are strongly peaked at the same fraction of slow particles. In the thermodynamic limit, they might diverge (as we would expect in a phase transition). A possible order parameter is f , the fraction of runs which contain a spanning cluster

The liquid-gas transition is also seen in a lattice simulation, with uniform heating, although without inelastic collapse.

5.4 Lattice gas simulation in 2D

This simulation consists of an $L \times L$ lattice with grains occupying a set fraction of the lattice sites. The liquid is “heated”. A grain is selected at random and given an impulse $v(t + \Delta t) = v(t) + f(t)\sqrt{r\Delta t}$ in a random direction (x or y). $f(t)$ is a random number between $-\frac{1}{2}$ and $\frac{1}{2}$ and r is a number proportional to the heating rate. If the grain has a $|v_{dir}t| > 1$ it attempts to move to the next lattice site. If the lattice site is empty, it succeeds. If the lattice site is filled, the two grains collide inelastically with coefficient of restitution η . When two particles i and j with initial velocities in the x direction of v_{ix} and v_{jx} collide in the x-direction their final (primed) velocities are

$$v'_{ix} = \frac{1}{2}(1 - \eta)v_{ix} + \frac{1}{2}(1 + \eta)v_{jx} \quad (9)$$

$$v'_{jx} = \frac{1}{2}(1 - \eta)v_{jx} + \frac{1}{2}(1 + \eta)v_{ix} \quad (10)$$

The y velocities remain unchanged.

The coefficient of restitution is less than one so particles approach each other more quickly than they recede after colliding. Therefore, particles which have collided spend more time in each other’s vicinities than particles which have not; e.g. they have correlated positions. One can simulate the two-point correlation function $g(x)$. In one dimension, $g(x) \sim x^{-\alpha(\eta)}$ where $\alpha(\eta)$ is a monotonically increasing function of η .

In two dimensions, there is a definite critical density above which liquid-gas coexistence begins to be observed. Large clusters of dense regions begin to appear, looking much as they did in the continuum simulation.

The susceptibility χ where

$$\chi = \phi^{-1} \sum_s^{\text{-largest}} s^2 n_s \quad (11)$$

n_s is the number of clusters of s particles. However, the sum omits the largest cluster. ϕ is the fraction of the sites occupied by particles. This definition makes sense because it is roughly proportional to the likelihood of a particle being in a cluster.

Two dimensional simulations suggest that χ peaks at some $\chi < \frac{1}{2}$ for a density ϕ above a critical ϕ . Furthermore, as one increases the size of the system, that is, as one approaches the thermodynamic limit, the peak begins to look like a divergence (see fig. 11).

6 Landau Theory

We begin with

$$\mathcal{L} = a_0 + a_1 T_g + a_2 T_g^2 + a_3 T_g^3 + S T_g + O(T_g^4) \quad (12)$$

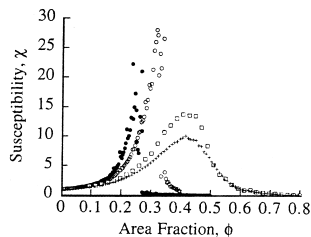


Figure 11: Susceptibility versus ϕ

We use the order parameter granular temperature, T_g , defined in equation 5. We define it slightly differently: we measure T_g in a small region and then talk about $T_g(r)$ because we want the order parameter to have a local value. In the Ising model, we have an order parameter η which is zero above T_c and non-zero below T_c . Granular temperature is really more of a disorder parameter because it is zero below a critical frequency and non-zero above f_c .

One can briefly compare this to the Ising model Landau Free Energy. In the Ising model, the energy of the system can be measured by measuring the temperature. In a granular system, energy comes from from physical moment. In experiments, energy is usually added to granular materials by shaking the material, the frequency of the shaking is taken to be a measure of the granular energy. E.g. frequency f is used instead of the temperature T .

Here, S is defined as the “inverse stickiness”. The lower S is, the stickier the grains are. We wanted a “field” term which would change the order of the system. In the Ising model, the field term is $-H\eta$. As H rises and the field term decreases the order in the Ising system rises. In the granular system, our field term is $-ST_g$. As the S increases and the field term decreases the amount of disorder in the system rises. Since we have a disorder parameter instead of an order parameter, it is appropriate that we look at the effect of the field term on the disorder of the system rather than the order.

Comparison of Ising and granular systems			
	Ising Model	Granular materials	
order parameter	M	T_g	disorder parameter
energy source	temperature	frequency	energy source
field term	-H M	-S T_g	field term

We take \mathcal{L} to third order in T_g because it’s the highest order with which which the algebra is easy.

\mathcal{L} must be minimized by T_g .

$$\frac{d\mathcal{L}}{dT_g} = a_1 + 2a_2T_g + 3a_3T_g^2 + O(T_g^3) \quad (13)$$

Since \mathcal{L} must be zero for $T_g = 0$ we get $a_1 = 0$.

Since \mathcal{L} must be a minimum at small T_g we say $a_3 \geq 0$. If $a_3 < 0$ then $\lim_{T_g \rightarrow \infty} \mathcal{L} = -\infty$, a global minimum.

Also, a_0 is the value of \mathcal{L} at high frequency shaking. It will most likely be analytic and slowly varying near F_c , the critical frequency. We subtract it off, and from now on consider $\mathcal{L} = \mathcal{L} - a_0$.

Symmetry considerations tell us nothing. T_g can not be negative so we can not eliminate any terms based on considering the relation between $\mathcal{L}(T_g)$ and $\mathcal{L}(-T_g)$.

We expand a_2 in f where $f = \frac{F-F_c}{F_c}$

$$a_2 = a_2^0 + a_2^1 f \quad (14)$$

We only take a_3 to 0th order in f . We are keeping terms of order T_g^3 and $T_g^2 f$ and ignoring all higher order terms.

Using these results we solve equation 13 for T_g .

$$T_g = \begin{cases} 0 & f < 0 \\ \frac{-a_2 f - \sqrt{a_2^2 f^2 + 3a_3 S}}{3a_3} & f > 0 \end{cases} = \frac{-(a_2^0 + a_2^1 f) f - \sqrt{(a_2^0)^2 + a_2^1 f)^2 + 3a_3 S}}{3a_3} \quad (15)$$

Experimentally, we know that $T_g = 0$ for $f = 0$ (see fig. 5). (We define our inverse stickiness field such that it was zero in the experiments.) Therefore, we set $a_2^0 = 0$. So for $f > 0$ and zero field we have

$$T_g = \frac{-a_2 f^2 - \sqrt{a_2 f^2 + 3a_3 S}}{3a_3} \quad (16)$$

From this we can read of the critical exponent $\beta = 1$ and $\delta = 2$.

6.1 Specific Heat

Our Landau free energy is now

$$\mathcal{L} = a_2 f T_g^2 + a_3 T_g^3 + O(T_g^4) \quad (17)$$

Using $C_v = -F \frac{\partial^2 \mathcal{L}}{\partial F^2}$ and our previously calculated value of T_g we obtain $C_v = \frac{8a_2^3 f(f+1)}{9a_3^2 F_c}$ for $f > f_c$. C_v is continuous as f goes to 0. Also, $\alpha = -2$.

Summary of Landau Results		
α	-2	reminder $C \sim f^{-\alpha}$
β	-1	$T_g \sim f^\beta$
δ	2	$T_g \sim S^{1/\delta}$

7 Conclusion

Granular phase transitions are understood to varying degrees. The liquid-gas transition has only been studied in the cooling regime. That is, the gas to liquid

transition has been studied, but the liquid to gas has not. The gas to liquid transition seems to be fairly well understood, probably in part because we know how the granular gas cools: it loses energy through inelastic collisions. The basic mechanism for gas to liquid transition is inelastic collapse. Particles leave the gas phase when they collide inelastically a large number of times and lose most of their energy. A great deal of interesting analytic and simulation work has been done on the gas to liquid transition. However, it is not clear how the liquid to gas transition would work.

The solid-fluid transition has been studied principally in the heating regime. Among solid fluid transitions, the avalanche induced transition is the most understood. However, the understanding of other solid-fluid transitions have not benefited greatly from this knowledge because the transition is so dependent on how energy is put into the system.

In sum, we understand how the granular materials cool, but not how they heat. There are many ways to heat a granular material, and they seem to result in quantitatively different transitions. For example, an avalanche is a first order solid-fluid transition but shaking is a second order transition. When we want to melt a conventional solid, we can either melt it or lower the surrounding pressure. However, the phase transition from solid to liquid is always the same order. The different orders we find for the solid-fluid transition probably means we have not found an appropriate order parameter. It could also mean that there are more phases and phase transitions than we are currently aware of.

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