Log-Periodic Scaling in Dynamical Systems

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Abstract

It has been proposed[7]–[9] that rupture processes in highly disordered media and earthquakes are similar to critical phenomena in that before they occur, long range correlations develop at many scales that lead to a cascade of events at increasingly larger scales. As a result, the precursory phenomena would be expected to follow power laws which are characteristic of critical phenomena. However, *log-periodic* scaling has been proposed as correction to this power law in the study of rupture processes of heterogeneous systems and earthquakes, and simulations have been carried out to show this correction does indeed exist[10]–[13]. This log-periodic correction-to-scaling, though a little surprising, could be actually derived from the point of view of Renormalization Group(RG), and may be usefully applied to predicting problems in natural sciences (earthquakes, volcanic eruptions, etc.)[10]–[11], and even possibly stock market crashes[14]–[16].

1 Introduction

Log-periodic oscillations in scaling systems preceding the critical point have been observed in several systems. Probably the first theoretical suggestion of their relevance to physics has been put forward by Novikov [1] to describe the influence of intermittency in turbulent flows. Preliminary experiments [2] seem to observe these oscillations on the time-dependent rate of convergence of structure functions. The interest in log-periodic oscillations has been somewhat revived after the introduction of the renormalization group theory of critical phenomena. The mathematical existence of such corrections has been discussed quite early in renormalization group solutions for the statistical mechanics of critical phase transitions [3]. However, these log-periodic oscillations, which amount to consider complex critical exponents, were rejected for translationally invariant systems, on the basis that a period implies the existence of one or several characteristic scales, which is forbidden in these ergodic systems in the critical regime. Complex exponents were therefore restricted to systems with discrete renormalization groups[10],[12].

An area where they were observed at length is low-dimensional dynamical systems. Indeed, the Feigenbaum/Coullet-Tresser sequence of subharmonic bifurcations to chaos can be understood from an asymptotically exact discrete renormalization group with a universal scaling factor, therefore leading to complex exponents and log-periodic oscillations around the main scaling as the dynamics converges to the invariant Cantor set measure at criticality [4]. Discrete scale invariance and complex exponents have also been discovered in dynamical stochastic processed in homogeneous systems when there exists a sufficiently strong intermittency, such as in models of diffusion in anisotropic quenched random media [5]. Another area involving discrete renormalization group is the study of Boolean delay equations. These are evolution equations for a vector of discrete variables, which model the evolution of biological and physical systems with threshold behavior and nonlinear feedback [6]. It has been found that, for boolean delay equations involving two time lags with an irrational ratio, the cumulative number of jumps J presents a superdiffuse behavior with log-periodic oscillations [6].

We will focus first on rupture in very heterogeneous media(including earthquake) of the existence of log-periodic scaling on the approach to major breakdown point. Then we will present several applications as well as simulations in support of the log-periodic scalings.

What is the nature of rupture associated with earthquake? An important working hypothesis, borrowed from Statistical Physics, is that earthquakes are similar to "critical" points: a progressive correlation develops leading to a cascade of events at increasingly large scales culminating in a larger earthquakes[7]. Failure in the crust can be thought of as a scaling-up process in which failure at one scale is part of the damage accumulation pattern at a larger scale. If a great earthquake can be viewed as a critical point, precursors of earthquakes should follow characteristic scaling laws. These power laws result naturally from the many-body interactions between the small cracks forming before the impending great rupture.

In general, the scaling law should only be observable very near the critical point, in the so-called "critical region", and take the form

$$\frac{d\epsilon}{dt} = k|t_f - t|^{-a} \tag{1}$$

where ϵ represents regional strain(or any other measure of seismic release), t_f is the time of failure, and k and a are constants. a is a critical exponent, which can often be interpreted geometrically in terms of some fractal underlying geometry. Integration of Eq.(1) yields

$$\epsilon = A + B|t_f - t|^m \tag{2}$$

where A, B and $m = 1 - a$ are constants.

The advantage of recognizing the above equations as universal scaling laws near a critical point is that we can proceed to derive the leading corrections to scaling, which are expected to be most important for any realistic prediction which must be made at the earliest time possible. It can be shown that these corrections take the form of a periodic function of $\log|t_f - t|$, which is superimposed on the power law in Eq.(2). This additional structure allows a more accurate prediction of t_f at a significantly earlier time in the sequence.

2 General Renormalization Group Framework for Critical Rupture and its Universal Leading Correction to Scaling

We first note that, an expression like Eq.(1) can be obtained from the solution of a suitable renormalization group(RG)[17]. The RG formalism, introduced in field theory and in critical phase transitions, amounts basically to the decomposition of the general problem of finding the behavior of a large number of interacting elements into a succession of simple problems with a smaller number of interacting elements, possessing effective properties varying with the scale of observation. Its usefulness is based on the existence of a scale invariance of self-similarity of the underlying physics at the critical point, which allows one to define a mapping between physical scale and distance from the critical point in the control parameter axis.

In the real-space version of RG, one translates literally in the real space the concept that rupture at some scale results from the aggregate response of an ensemble of ruptures at a smaller scale. In the earthquake problem, the seismic release rate $\frac{d\epsilon}{dt}$ at a given time t is related to that at another time t' by the following transformations

$$x' = \phi(x) \tag{3}$$

$$F(x) = g(x) + \frac{1}{\mu}F(\phi(x)) \tag{4}$$

where $x = t_f - t$, t_f is the time of global(regional) failure for the region under consideration, ϕ is called the RG flow map, $F(x) = \epsilon(t_f) - \epsilon(t)$ such that $F = 0$ at the critical point and μ is a constant describing the scaling of seismic release rate upon the discrete time rescaling(3). The function $g(x)$ represents the non-singular part of the function $F(x)$. We assume that the function $F(x)$ is continuous and that $\phi(x)$ is differentiable.

The critical point(s) is(are) described mathematically as the time(s) at which $F(x)$ becomes singular, i.e., when there exists a finite k^{th} derivative $\frac{d^kF(x)}{dx^k}$ which becomes infinite at the singular point(s). The formal solution of Eq.(4) is obtained by considering the following definitions:

$$f_0(x) = g(x) \tag{5}$$

and

$$f_{n+1}(x) = g(x) + \frac{1}{\mu}f_n[\phi(x)], n = 0, 1, 2, \cdots \tag{6}$$

It is easy to show by induction that

$$f_n(x) = \sum_{i=0}^{n} \frac{1}{\mu^i} g[\phi^{[i]}(x)], n > 0 \tag{7}$$

Here, we have used superscripts in the form "n" to designate composition, i.e., $\phi^{[2]} = \phi[\phi(x)]$. It naturally follows that

$$\lim_{n\to\infty} f_n(x) = F(x) \tag{8}$$

assuming that it exits.

The power of RG analysis is to reconstruct the nature of the critical singularities from the embedding of scales, i.e. from the knowledge of the non-singular part $g(x)$ of the observable and the flow map $\phi(x)$ describing the change of scale. The connection between this formalism and the critical point problem stems from the fact that critical points correspond to the unstable fixed points of the RG flow $\phi(x)$. Indeed, as in standard phase transitions, a singular behavior emerges from the infinite sum of analytic terms, describing the solution for the observable $F(x)$, if the absolute value of the eigenvalue λ defined by

$$\lambda = \frac{d\phi}{dx}|_{x=\phi(x)} \tag{9}$$

becomes larger than 1, in other words, if the mapping ϕ becomes unstable by iteration at the corresponding(critical) fixed point. In this case, the i^{th} term in the series for the k^{th} derivative of $F(x)$ will be proportional to $(\frac{\lambda^k}{\mu})^i$ which may become larger than the unit radius of convergence for sufficiently large k, hence the singular behavior.

Thus, the qualitative behavior of the critical points and the corresponding critical exponents can be simply deduced from the structure of the RG flow $\phi(x)$. If $x = 0$ denotes a fixed point($\phi(0) = 0$) and $\phi(x) = \lambda x + \cdots$ is the corresponding linearized transformation, then the solution of Eq.(4) close to $x = 0$ is given by Eq.(2), i.e., $F(x) \sim x^\alpha$, with α solution of

$$\frac{\lambda^\alpha}{\mu} = 1 \tag{10}$$

which yields $\alpha = \frac{\log \mu}{\log \lambda}$.

To get the leading correction in the critical region, we assume that $F_0(x) \sim x^\alpha$ is a special solution, then the general singular solution $F(x)$ is related to $F_0(x)$ in terms of an *apriori* arbitrary periodic function $p(x)$, with a period $\log \mu$, as

$$F(x) = F_0(x)p(\log F_0(x)) \tag{11}$$

to get this expression, we have neglected the non-singular term $g(x)$ in Eq.(4). The solution of Eq.(11) can then be checked by inserting it into the equation $F(\lambda x) = \mu F(x)$ which, because of Eq.(10), is also obeyed by F_0. Then we get the periodicity requirement

$$p(\log \mu + \log F_0(x)) = p(\log F_0(x)) \tag{12}$$

as asserted. Since $F_0(x) \sim x^\alpha$, this introduces a periodic(in $\log x$) correction to the dominating scaling Eq.(2) which amounts to considering a complex critical exponent α, since $Re[x^{\alpha' + i\alpha''}] = x^{\alpha'} cos(\alpha'' \log x)$ gives the first term in a Fourier series expansion of Eq.(11). Note that one can get directly this complex critical exponent by noting that Eq.(10) can be rewritten as $\frac{\lambda^\alpha}{\mu} = e^{i2\pi n}$, where n is an integer. Its solutions reads

$$\alpha = \frac{\log \mu}{\log \lambda} + i\frac{2\pi n}{\log \lambda} \tag{13}$$

which allows us to recover exactly the previous Fourier series expansion with $\alpha'' = \frac{2\pi n}{\log \lambda}$.

Expression(11) thus introduces universal oscillations decorating the asymptotic power law(2). The mathematical existence of such corrections have been identified quite early[3] in RG solutions for the statistical mechanics of critical phase transitions, but have been rejected for translationally invariant systems, since a period(even in a logarithmic scale) implies the existence of one or several characteristic scales, which is forbidden in these ergodic systems in the critical regime. Another equivalent point of view for understanding these Log-periodic correction to scaling is to realize that the RG written with Eq.(3) and Eq.(4) is discrete, i.e., one goes from a time t to another time t' which is at a finite(and not an infinitely small) interval from t. This captures the fact that damage and precursory phenomena occur at particular discrete times and not in a continuous fashion, and these discontinuities reflect the localized and threshold nature of the mechanics of earthquakes and faulting. It is this "punctuated" physics which gives rise to the existence of scaling precursors modeled mathematically by the Log-periodic correction to scaling.

Finally, we can modify Eq.(2) as

$$\epsilon(t) = A + B(t_f - t)^m [1 + C\cos(2\pi \frac{\log(t_f - t)}{\log \lambda} + \Psi)] \tag{14}$$

3 Applications

3.1 Scaling laws for earthquake and fracture of heterogeneous materials and rock

It was shown[10] that Eq.(14) can provide accurate predictions for the time t_f at which some large earthquakes have already occurred. For example, Eq.(14) predicts that the Loma Prieta earthquake in northern California, which had a magnitude of $6.7 - 7.1$, should occur in $1989.9 + -0.8$; the earthquake actually occured on Oct. 17, 1989. See Fig.(1).

The log-periodic corrections to scaling also implies the existence of a hierarchy of characteristic times t_n, determined from the equation $2\pi\frac{\log(t_f-t_n)}{\log\lambda} + \Psi = n\pi$, which yields $t_n = t_f - T\lambda^{\frac{3}{2}}$, with $T = \lambda^{-\frac{\Psi}{2\pi}}$. For the Loma-Prieta earthquake, $\lambda \approx 3.3$ and $T \approx 1.3$ years. And what is expected to be universal are the ratios $\frac{t_f-t_{n+1}}{t_f-t_n} = \lambda^{\frac{1}{2}}$. These time scales could reflect the characteristic relaxation times associated with the coupling between stress(or strain) and between the brittle and lower ductile crusts.

Computer simulations have also been carried out recently[13] to study the fracture of heterogeneous solids. There they studied an $L*L$ triangular network with a periodic boundary condition in one direction, each site of which is characterized by the displacement vector $\mathbf{u}_i = (u_{ix}, u_{iy})$, with nearest-neighbor sites connected by springs, where each spring represents a portion of rock at a small scale.The case of brittle fracture is considered for which a linear approximation is valid up to a threshold(defined below). The displacement $\mathbf{u}_i$ is computed by minimizing the elastic energy E with respect to $\mathbf{u}_i$, where

$$E = \frac{\alpha}{2} \sum_{<ij>} [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{R}_{ij}]^2 g_{ij} + \frac{\beta}{2} \sum_{<jik>} (\delta\theta_{jik})^2 g_{ij} g_{ik} \qquad (15)$$

Here α and β are the central and bond-bending or angle-changing force constants, respectively, $\mathbf{R}_{ij}$ is a unit vector from site i to site j, g_{ij} is the elastic constant of the spring between i and j, and $< jik >$ indicates that the sum is over all triplets in which the bonds $j - i$ and $i - k$ form an angle whose vertex is at i. A threshold value l_c for the length of a spring is introduced, which is selected from a probability distribution. Two types of threshold distributions were used. One was a power law,

$$f(l_c) = (1 - \gamma) l_c^{-\gamma} \qquad (16)$$

with $0 < \gamma < 1$. Thus, for $\gamma \approx 0$ one has a narrow distribution of the threshold, whereas for $\gamma \approx 1$ one has a very broad and heterogeneous distribution. However, this distribution is completely random and provides no correlations between various regions of the system.

The failure process is initiated by applying a fixed external strain to the network in a given direction, calculating $\mathbf{u}_i$'s, and breaking all the springs whose length have exceeded their critical threshold l_c, where each broken spring represents a microcrack, and cumulative released elastic energy ΔE by fracturing of the system is also calculated. Fig.(2) shows the results with the random distribution of the thresholds for a single realization, where both the simulation results and the fit to Eq.(14) are presented($\gamma = 0.5$). Fig.(3) presents the same type of data as those in Fig.(2), except that now the results represent the average of several realizations of the system. As can be seen, the oscillatory fluctuations do not die out when the behavior of the system is averaged over many realizations. Correlated distribution of the thresholds is believed to be more relevant to heterogeneous rock at large length scales, and as can be seen from Fig.(4), Eq.(14) still provides an accurate fit to the data.

The existence of such corrections to the scaling of the released elastic energy may be explained as follows. The rock first develops isolated microcracks which are nucleated in the weak regions. As the applied strain increases, more weak regions develop cracks, while the stronger regions remain relatively intact. After some time the microcracks join and a fractal fracture pattern emerges which explains the existence of the power law in Eq.(2). At the same time, because the weaker regions have already failed, for a range of the applied strain no major crack is formed, and the released elastic energy varies little until a value of the strain S is reached at which a number of cracks can form, and therefore the released energy increases sharply. This is repeated in cycles that are, similar to the earthquake data, of increasingly smaller size in the applied strain(time interval in the earthquake data), but larger in the released energy, since as more fractures are formed one nears the critical region in which more and more cracks are created with an increasingly smaller change in the applied strain, until the sample-spanning fracture network is formed and global failure occurs. Thus, we may view seismicity in a region as a sequence of fracturing cycles, where each cycle represents a progressive cooperative stress build up and crack nucleation that culminate in some sort of a critical point-the formation of the sample-spanning fracture network-which is characterized by global failure in the form of a large earthquake. While Saleur, Sornette[12] have suggested that the existence of *discrete invariance* is essential for having log-periodic corrections. The results above by Sahimi[13] indicate that the interplay between the heterogeneties of rock and the stress field generates *dynamically* such discrete scale invariance, and does not have to be present in the rock structure itself.

In general, log-periodic corrections may be used for predictions of earthquakes[14] and fractures of heterogeneous materials[13]: one fits the rate of acoustic emissions, a measurable quantity, to an equation like (14) to predict the time t_f at which earthquake or fracture of material will occur. Log-periodic corrections have also been shown to be important in the critical properties of spin systems, if they are defined on a hierarchical lattice[18].

3.2 Scaling laws for large stock market crashes

Sornette and Johansen[16] proposed that large stock market crashes are analogous to critical points with log-periodic correction to scaling. Yet this time a log-frequency shift over time is introduced besides the log-periodic corrections, which can be obtained as follows.

First, we notice that the solution(14) of the RG Eq.(4) together with Eq.(3) and the linear transformation $\phi(x) = \lambda x$ can be rewritten as

$$\frac{dF(x)}{d\log x} = \alpha F(x) \tag{17}$$

stating simply that a power law is nothing but a linear relationship when expressed in the variables $\log F(x)$ and $\log x$. Eq.(17) can be interpreted as a bifurcation equation for the variable F as a function of a fictitious "time"($\log x$) as a function of the "control parameter" α. When $\alpha > 0$, $F(x)$ increases with $\log x$ while it decreases for $\alpha < 0$. The special value $\alpha = 0$ separating the two regimes corresponds to a bifurcation[19]. Thus the general reduction theorem[19] tells us that the structure of the equation for F close to the bifurcation can only take a universal $non-linear$ form given symmetries. Introducing the amplitude B and phase ψ of $F(x) = Be^{i\psi(x)}$, the only symmetry to be used is the fact that a global shift of the phase should keep the observable constant under a global change of units. This implies the following expansion:

$$\frac{dF(x)}{d\log x} = (\alpha + i\omega)F(x) + (\eta + i\kappa)|F(x)|^2 F(x) + O(F^5) \tag{18}$$

where $\alpha > 0$, ω, η and κ are real coefficients and $O(F^5)$ means that higher order terms are neglected. Such expansion are known in the physics literature as Landau expansions[17]. This expression represents a non-trivial addition to the theory, constrained uniquely by symmetry laws. The solution of Eq.(18) reads

$$B^2 = B_\infty^2 \frac{(\frac{x}{x_0})^{2\alpha}}{1+(\frac{x}{x_0})^{2\alpha}}, \tag{19}$$

$$\psi = \omega \log \frac{x}{x_0} + B_\infty^2 \frac{\kappa}{2\alpha} \log(1+(\frac{x}{x_0})^{2\alpha}) \tag{20}$$

and leads to the following modification of Eq.(14):

$$I(\tau) = A + B\frac{(\tau_c-\tau)^\alpha}{\sqrt{1+(\frac{\tau_c-\tau}{\Delta t})^{2\alpha}}}[1+C\cos(\omega\log(\tau_c-\tau)+\frac{\Delta\omega}{2\alpha}\log(1+(\frac{\tau_c-\tau}{\Delta t})^{2\alpha}))] \tag{21}$$

Two new effects are predicted by Eq.(21)

- There is a saturation of the function $I(\tau)$ far from the critical point;
- The log-frequency shifts from $\frac{\omega+\Delta\omega}{2\pi}$ to $\frac{\omega}{2\pi}$, when approaching the time of the crash.

Eq.(21) is used to fit the Dow Jones index prior to the 1929 crash and the S&P500 index prior to the 1987 crash, both starting approximately 8 years prior to the crash. Here the logarithm of the index is used for the measurement. Fig.(5) and Fig.(6) show the best fit. It can be seen that the general trend of the data is well-captured by the proposed relation over more than 7 years. In order to quantify this statement, the relative error of the fit to the data has been calculated and found to be $\leq 10\%$ on the entire time interval. In Fig.(6), the thin line represents the best fit with Eq.(21) over the whole time interval, while the thick line is the fit by Eq.(14) on the subinterval from July 1985 to the end of 1987 as done in Ref.[14] but is represented on the full time interval starting in 1980. The comparison with the thin line allows one to visualize the frequency shift described by Eq.(21). The results are also quantified by the observation that the value of the exponent α for the two great crashes are quite close to each other, namely $\alpha_{1929} = 0.63$ and $\alpha_{1987} = 0.68$, which is in agreement with the universality of the exponent α predicted from the renormalization group theory.

However, the above results do not justify that stock market crashes can be predicted exactly ahead of the time. Feigenbaum[15] argued that, a crash can occur over a day(e.g. the 1987 crash), or over months(e.g., the 1967 crash), and thus the precise relationship between the critical time t_f involved in a log-periodic fit and the time of the crash requires further study. So does the issue of the accuracy to which the critical time and the time of the crash can be predicted by monitoring the index data for log-periodic oscillations. Indeed, were the trader to learn how to decipher and use the information provided by stock market prices, they would act on it and on the knowledge that others act on it and the crashes would probably not happen. Such seemingly paradox suggests a weaker form of the "weak efficient market hypothesis"[20], according to which the market prices contain, in addition to the information generally available to all, subtle information formed by the global market that most or all traders have not yet learned to decipher and use. The market as a whole can exhibit an "emergent" behavior not shared by any of its constituent, and we have in mind the process of the emergency of intelligent behavior at a macroscopic scale that individuals at the microscopic scale have not idea of.

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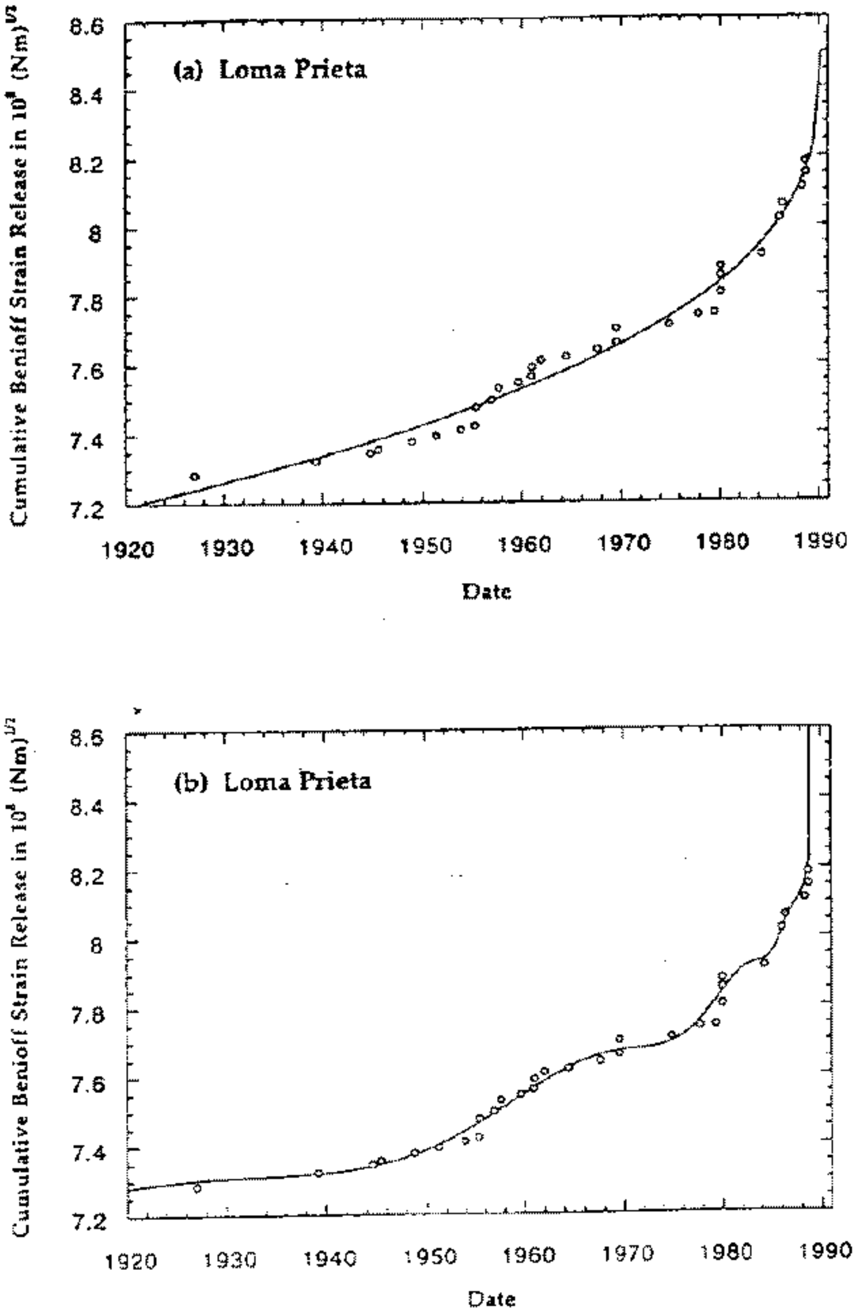

Figure 1: Cumulative Benioff strain released by magnitude 5 and greater earthquake in the San Francisco Bay area prior to the 1989 Loma Prieta earthquake. In (a), the data have been fit to the power law Eq.(2). In (b), the data have been fit to Eq.(14) which includes the first order correction to scaling.

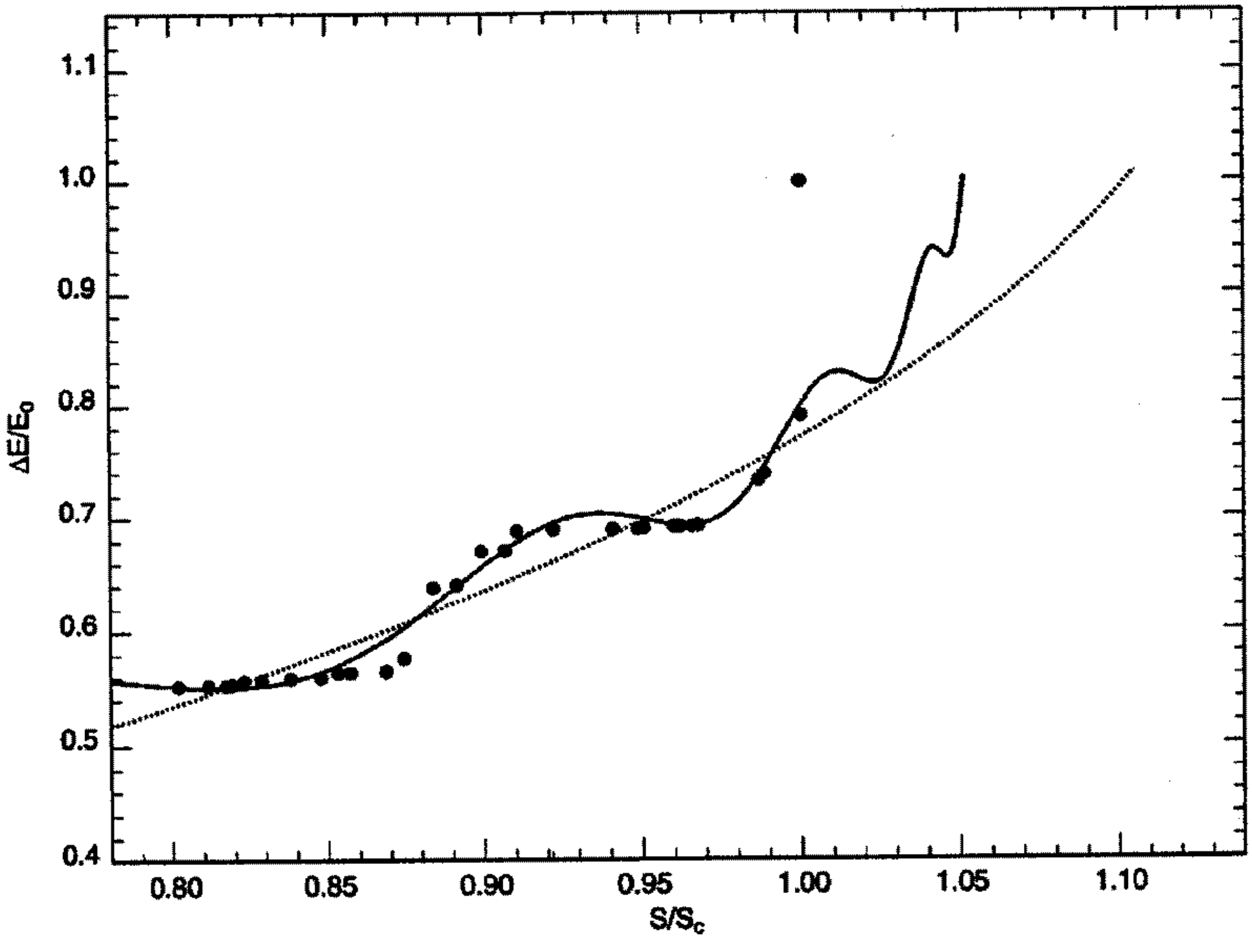

Figure 2: Variations of the cumulative elastic energy ΔE released during fracturing(normalized by the energy E_0 of the unfractured system) with the external strain S, normalized its value S_c just before the global failure, for a single realization of a random system. The dashed curve shows the fit of the data(circles) by a simple power law, whereas the solid curve shows the fit with log-periodic corrections with the power law.

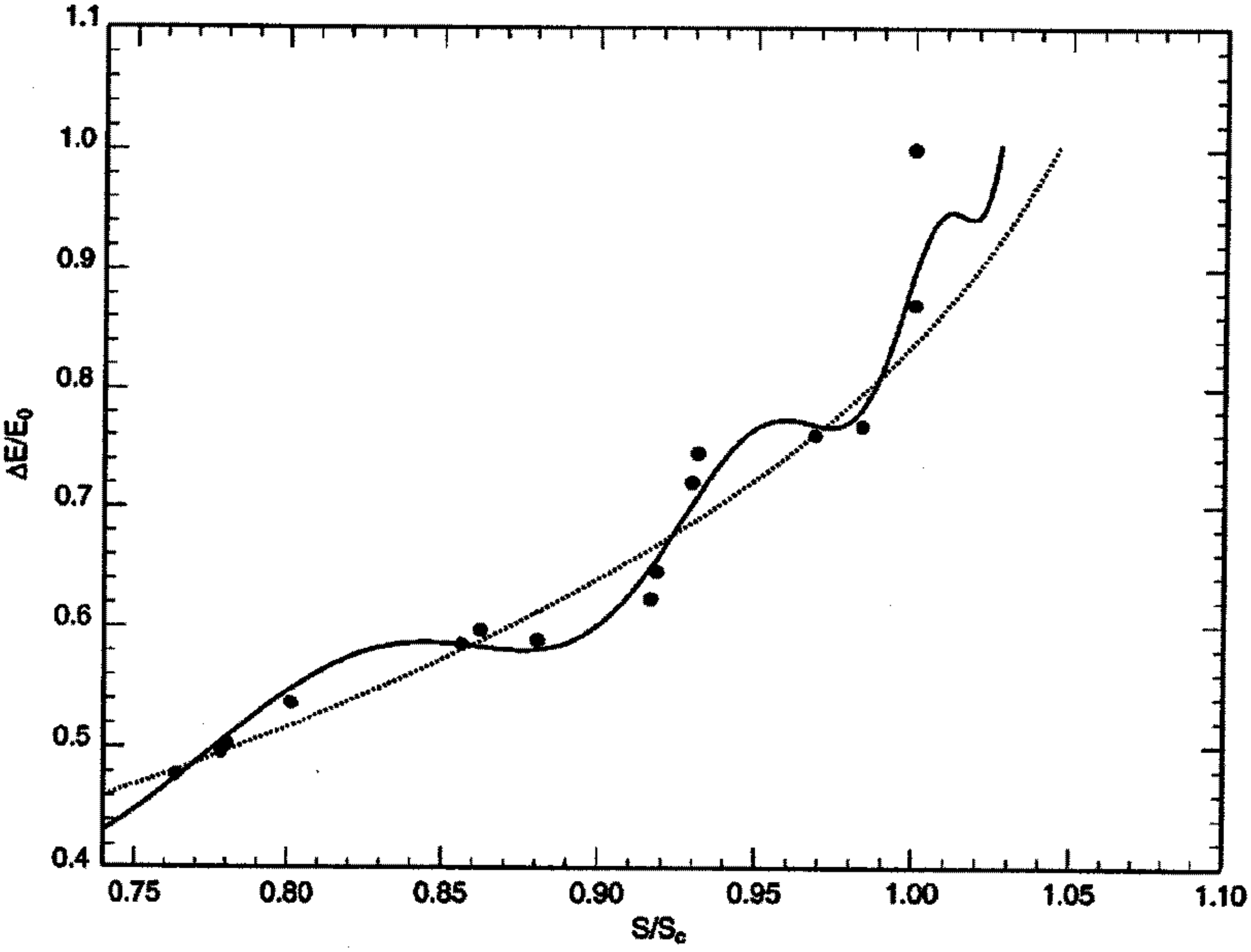

Figure 3: The same as in Fig.(2), but averaged over several realization of the system

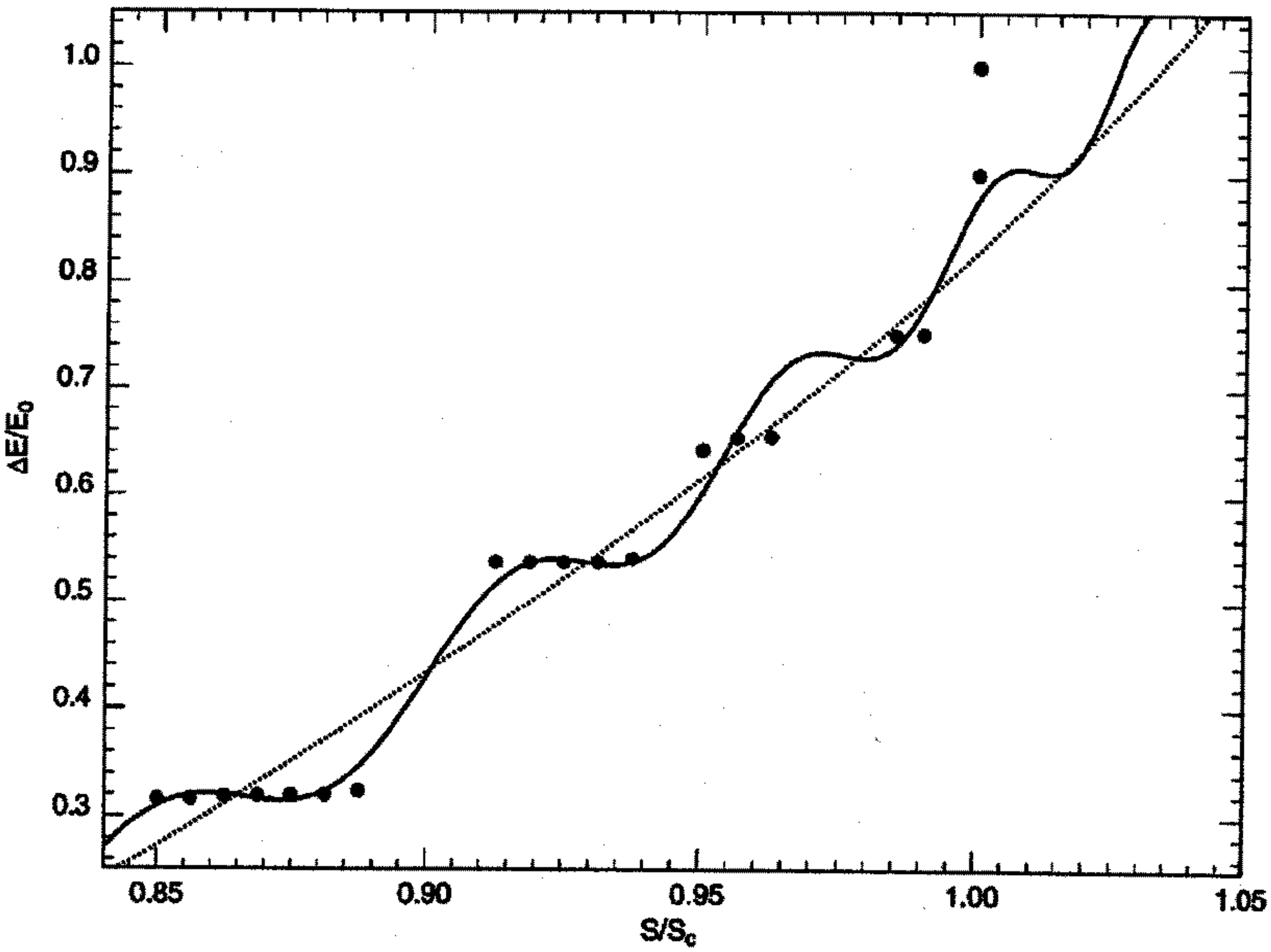

Figure 4: The same as in Fig.(2), but for a correlated system

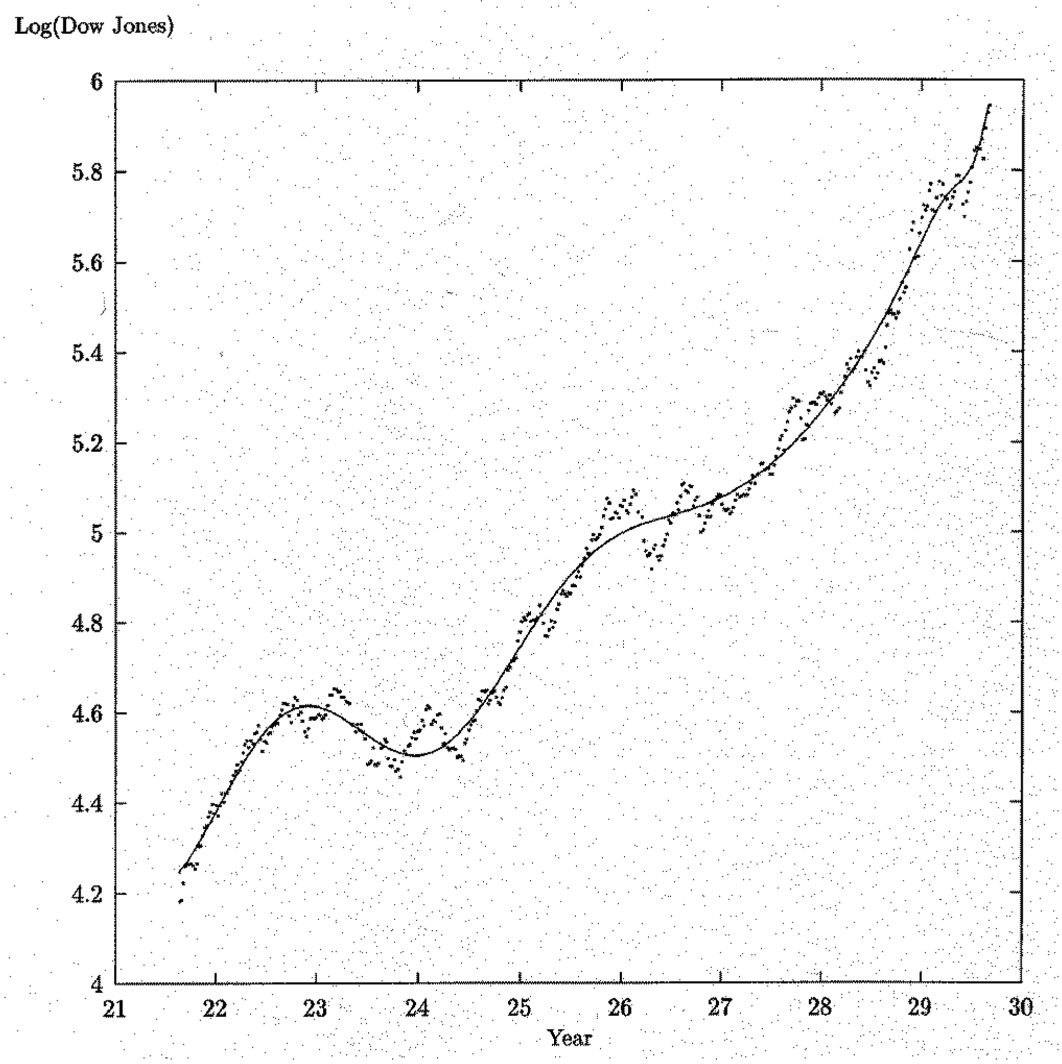

Figure 5: Time dependence of the logarithm of the Dow Jones stock exchange index from June 1921 to September 1929 and the best fit by Eq.(21). The crash of October 23, 1929 corresponds to 1929.81 decimal year. The parameters of the fit are: r.m.s.=0.041, t_c=1929.84 year, $\alpha = 0.63$, $\omega = 5.0$, $\Delta\omega$=-70, $\Delta t = 14$ years, A=61, B=-0.56, C=0.08.

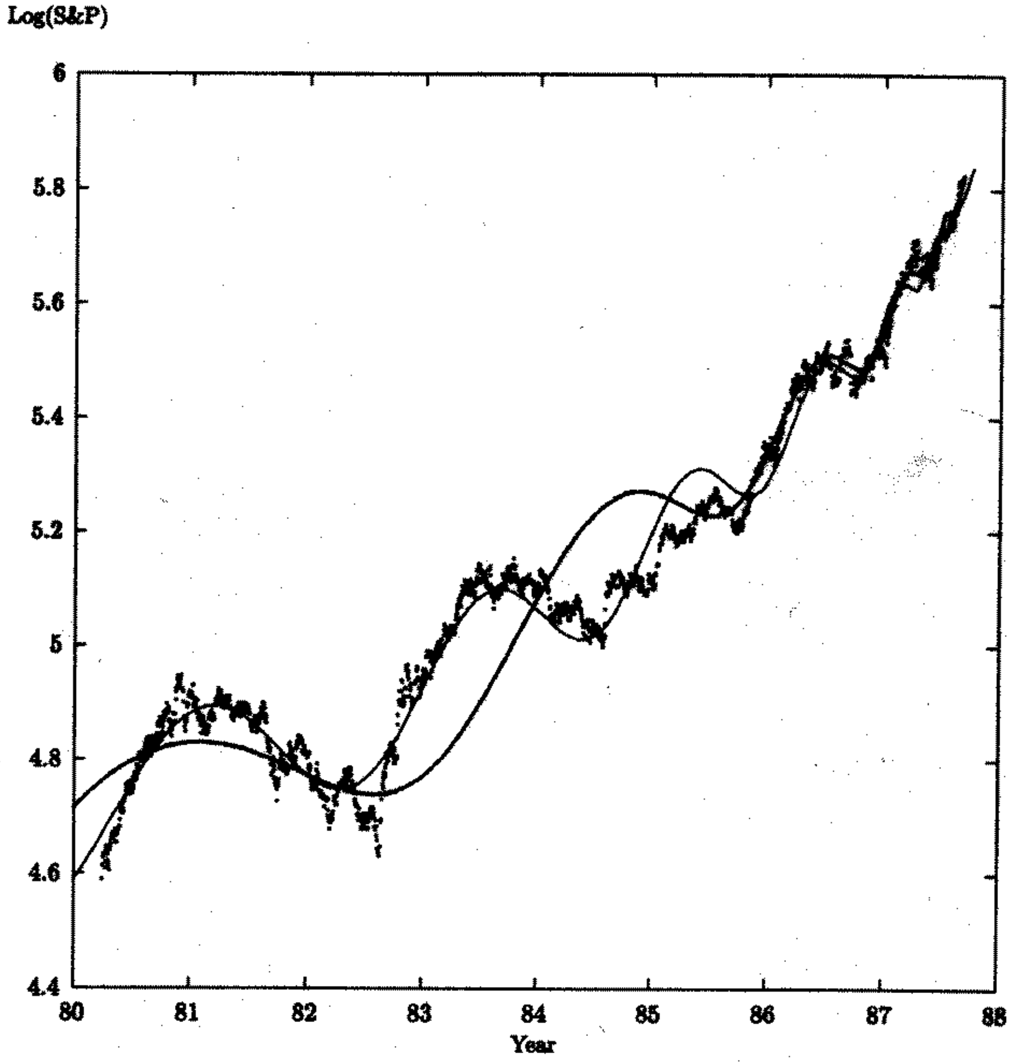

Figure 6: Time dependece of the logarithm of the New York stock exchange index S&P500 from January 1980 to September 1987 and the best fit by Eq.(21)(thin line). The Crash of October 14, 1987 corresponds to 1987.78 decimal year. The thin line represents the best fit with equation parameter of the fit are: r.m.s.=0.043, $t_c = 1987.81$ year, $\alpha = 0.68$, $\omega = 8.9$, $\Delta\omega = 18$, $\Delta t = 11$ years, $A = 5.9$, $B = -0.38$, $C = 0.043$. The thick line is the fit by Eq.(14) on the subinterval from July 1985 to the end of the 1987 and is represented on the full time interval starting in 1980. The parameters of this fit with Eq.(14) are: r.m.s.=6.2, $t_c = 1987.74$ year, $\alpha = 0.33$, $\omega = 7.4$, $A = 412$, $B = -165$, $C = 0.07$. The comparison with the thin line allows one to visualize the frequency shif described by Eq.(21).