Self-Organized Criticality

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Abstract

The concept of self-organized criticality (SOC) has been evoked by Bak, Tang and Wiesenfeld (BTW) as a possible unifying framework to describe a vast class of dynamically driven systems which evolve spontaneously toward a critical stationary state with no characteristic time or length scale. In this essay we will discuss SOC models that have been studied and to what extent they are related to real physical systems. We will also present recently developed analytical approaches, among them a renormalization group scheme, the Dynamically Driven Renormalization Group.

1 Introduction

Under the ambitious title "Self-Organized Criticality: An Explanation of 1/f noise " Bak, Tang and Wiesenfeld introduced the concept of self-organized criticality. They argued that dynamical systems with extended spatial degrees of freedom naturally evolve into barely stable states far from equilibrium which exhibit power law behaviour in their statistical properties. Unlike in ordinary equilibrium critical phenomena this is achieved without tuning of parameters of the system, such as the reduced temperature in the equilibrium case, to a critical value. The critical point in systems viewed as self-organized critical is an attractor reached for any initial condition.

Although BTW's original claim that SOC is the underlying mechanism behind the phenomena of 1/f noise and their spatial counterparts, fractals, could not be maintained, their paper triggered intense research activity. Numerous real systems were suggested to exhibit SOC behaviour, among them such different systems as earthquakes, superconductors and biological evolution. Various models were introduced to study SOC, the most extensively studied being the sandpile model (which was the first SOC model introduced by BTW), the forest fire model and the Olami-Feder-Christensen earthquake model. These models are cellular automata and motivated by real physical systems. They are usually defined on a d-dimensional lattice. Their dynamics is governed by an algorithm with discrete time steps and a discrete or continuous dynamical variable whose evolution is studied. More precisely, the systems are driven by some external field, and whenever the dynamical variable at some site exceeds a threshold value a relaxation event takes place. This relaxation may be confined to a small region in space or cause "large events" (avalanches in SOC terminology) of the size of the system.

There have also been several analytical approaches to come to a better understanding of SOC. For the class of so-called Abelian sandpiles, a matrix formalism has yielded some exact results. There exists a variety of mean field desriptions of SOC models. The Dynamically Driven Renormalization Group (DDRG) method seems to be the most promising tool in describing SOC and puts it in the more general context of nonequilibrium systems with a stationary state.

2 Models

2.1 Sandpile Model

We begin with the introduction of scaling relations and critical exponents for the sandpile model. In the study of sandpile models three distributions are of interest. Let

l the linear size of an avalanche

s the total energy release (total number of sites that must become relaxed before all sites are undercritical)

t the lifetime of an avalanche (number of updates needed to make all sites undercritical).

We consider

$$
\begin{aligned}
P(l) &= \int ds \int dt P(l,s,t) \\
P(s) &= \int dl \int dt P(l,s,t) \\
P(t) &= \int dl \int ds P(l,s,t)
\end{aligned}
$$

The sandpile model is considered critical because these distributions are observed to obey the following power laws:

$$
\begin{aligned}
P(l) &\sim l^{-\lambda} \\
P(s) &\sim s^{-\tau} \\
P(t) &\sim t^{-\alpha}
\end{aligned}
$$

In a critical state we also expect scale-independent relationships between the variables that characterize the avalanches. With the conditional expectation value defined as $E[X|Y=y] = \sum_x xP(X=x|Y=y)$ we define critical exponents as follows

$$
\begin{array}{lll}
E[S|T=t] \sim t^{\gamma_1} & E[S|L=l] \sim l^{\gamma_2} & E[T|L=l] \sim l^{\gamma_3} \\
E[T|S=s] \sim s^{1/\gamma_1} & E[L|S=s] \sim s^{1/\gamma_2} & E[L|T=t] \sim t^{1/\gamma_3}
\end{array}
$$

Using the identity $\int dy E[X|Y = y]P(Y = y) = \int dz E[X|Z = z]P(Z = z)$ which is valid for any set of three stochastic variables X,Y,Z one obtains after some simple algebra

$$\alpha = 1 + \frac{\lambda - 1}{\gamma_3} \quad , \quad \tau = 1 + \frac{\lambda - 1}{\gamma_2}$$

Most of the analytical and numerical work in the literature has been done in 2d. One result is that the avalanches are compact, i.e. $s \sim l^2$, so $\gamma_2 = 2$. The exponent γ_3 is usually denoted z and is called the dynamical critical exponent. Thus in 2d we have

$$\lambda = 1 + 2(\tau - 1) \quad , \quad \alpha = 1 + \frac{2(\tau - 1)}{z}$$

and we can define τ and z as the independent critical exponents. These relations are in agreement with numerical simulations.
The most studied sandpile model is the so-called critical height sandpile model. It is formulated as a cellular automaton with an integer height variable h_i on each site of a d-dimensional lattice (the dynamical variable). We will consider the case 2d here. Particles are added randomly (the "driving field") and the addition of a particle at site i increases its height by 1:

$$h_{i,j} \rightarrow h_{i,j} + 1.$$

The stability of a "sand column" depends on a preassigned critical height. If a site exceeds this critical value, n particles are distributed to (usually) nearest neighbours ("relaxation event" , "toppling"). For a nondirected version one chooses $n = 4$ so that for an active state i,j

$$\begin{aligned} h_{i,j} &\rightarrow h_{i,j} - 4 \\ h_{i\pm1,j\pm1} &\rightarrow h_{i\pm1,j\pm1} + 1 \end{aligned}$$

i.e. the dynamical rule is isotropic. Through this update nearest neighbours can become overcritical themselves and the above rule is applied again until all sites are undercritical again. Another particle is added *after* the avalanche has stopped. This means a separation of time scales: the relaxation is infinitely small compared with the driving rate. We will talk more about this important point later.

We assume open boundary conditions, i.e. particles crossing the boundary are lost. The system evolves into a statistically stationary state after a transient time; the average number of added particles equals the number of particles lost at the boundaries. This is the critical state. Results of com-

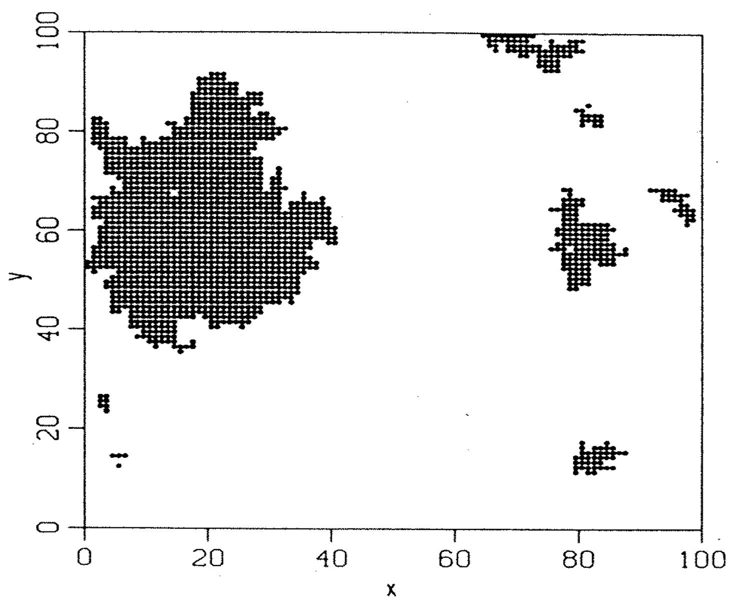

Figure 1: Critical state of minimally stable clusters.

puter simulations that BTW made are shown in Figures 1, 2 and 3. Figure 1 shows typical domain structures that resulted from several local perturbations. Each cluster is triggered by a single perturbation. One can clearly see that at the critical state there are events of all length scales. The distribution of cluster (avalanche) sizes in Figure 2 clearly exhibits power law behaviour . The deviation from a straight line at large avalanche sizes comes from finite size effects. The distribution of avalanche lifetimes (Figure 3) also follows a power law . For long lifetimes deviations occur because of a lack of reliable data.

The original claim of BTW was that SOC is the underlying mechanism behind the abundant appearance of fractal structures and the ubiquitous

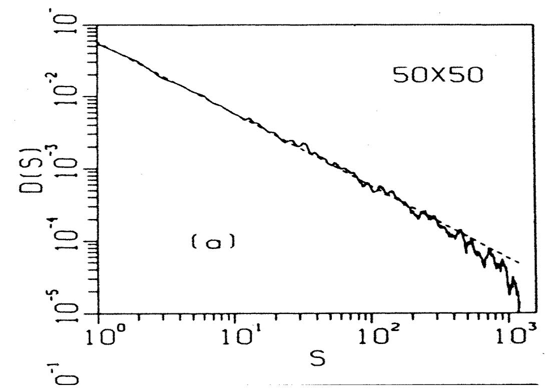

Figure 2: Avalanche size distribution.

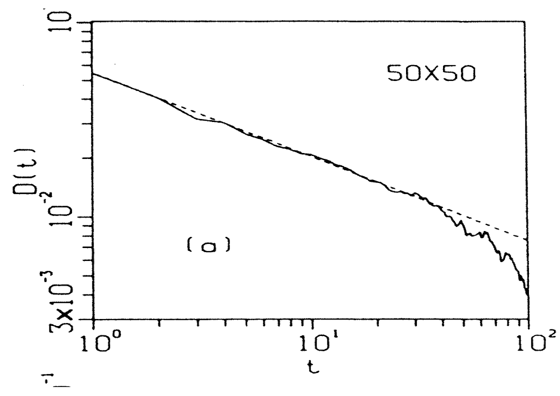

Figure 3: Avalanche lifetime distribution.

$1/f$ noise. $1/f$ noise refers to a power spectrum[1] of the form $f^{-a}, a \approx 1$. This essentially means long-time correlations and was seen as the temporal counterpart of fractal structures that are related with large correlations in space. BTW considered the power spectrum of the total number of topplings at time t, i.e. the growth rate of avalanches and obtained values of a of roughly 1 in 2d and 3d by means of computer simulations. Shortly later, however, it was shown that their analysis had not been quite right and that the power spectrum is rather $1/f^2$ noise.

[1]The power spectrum of a time dependent signal is defined in terms of the square amplitude of the Fourier transform of this signal

Before we turn to analytical approaches to SOC we will shortly discuss two other models that have been extensively studied and address the issue of the relevance of these models to real physical systems.

2.2 Forest Fire Model

The forest fire model (FFM) is defined on a d-dimensional cubic lattice. Each of the lattice sites either contains a green tree or a burning tree or is empty. The update algorithm is as follows:

1. A site occupied by a burning tree becomes an empty site in the succeding time step.

2. A green tree becomes a burning tree if one or more of its nearest neighbour sites contain a burning tree.

3. An empty site becomes occupied by a green tree with probability p (growth rate) in each time step.

4. A green tree that is not neighbour to burning sites catches fire spontaneously with probability f (lightning rate) in each time step.

Periodic boundary conditions are assumed.
The fourth rule is crucial: without it the system would not exhibit critical behaviour. Moreover, numerical simulations and analytical analysis show that criticality only occurs in the double limit $p \to 0, f/p \to 0$. Thus, because it has relevant parameters, the forest fire model is rather critical in the ordinary sense. We will come back to this point later.

Numerical simulations in dimensions 1-8 show power law behaviour in the limits mentioned above. Above d=6 the scaling exponents do not change with dimension, indicating that 6 is the upper critical dimension.

A mean field description of the FFM maps it into percolation. Thus one can use results of percolation theory, namely that $d_c = 6$ as was suggested by simulations.

2.3 Earthquake Model

The Olami-Feder-Christensen (OFC) earthquake model was introduced as a simple representation of the dynamics of the Burridge-Knopoff spring-block model which models earthquakes. The dynamical variable can be viewed as a local force or stress or just as energy. An important difference to the critical height sandpile model is that the OFC model includes nonconservative updates (not only for the boundaries, but also for the bulk sites) in its algorithm. Criticality was considered to occur only in conservative systems and therefore the criticality of the OFC model came as a surprise. The model is defined on a d-dimesional lattice. We assign to each site a real dynamical variable E_i. During each time step

$$E_i \to E_i + \nu. \tag{1}$$

As in the case of the sandpile model the driving is switched on only as long as all $E_i < E_c$, with E_c being some threshold value. When a site becomes unstable, the following local relaxation rule is applied:

$$E_i \geq E_c \Rightarrow \begin{cases} E_i \to 0 \\ E_{nn} \to E_{nn} + \alpha E_i \end{cases}$$

where nn denotes nearest neighbours and α is the conservation level. Thus the energy lost in each time step is

$$E_{dis} = (q_i \alpha - 1) E_i \tag{2}$$

where q_i is the coordination number of the ith lattice site, being $q_c = 2d$ for a bulk site and assuming a different value at the boundary in an obvious way. Thus $\alpha < 1/q_c$. The update is iterated until all sites are undercritical again, just as in the sandpile model. We consider d=2. The OFC model reaches a stationary critical state after a very long transient time (about 10^8 avalanches for a system size 10×10). The exponent τ depends on the conservation level α. The behaviour is critical in the range $\alpha_c < 1/q_c$ where α_c is a small but nonzero cut-off value.
The critical state only exists for open boundary conditions. Periodic boundary conditions lead to a periodic stationary state.
The behaviour of τ_α as a function of α is unclear. For small system sizes $(25 \leq L \leq 45)$ and values of α not too far form 1/4 Christensen and Olami found their data to be consistent with the finite size scaling form

$$P_\alpha(s, L) = L^{-\beta} g_\alpha \left(\frac{s}{L^\nu} \right). \tag{3}$$

where g_α is an α-dependent scaling function common to all system sizes in the above range. This can be written as

$$P_\alpha(s,L) = s^{-\frac{\beta}{\nu}} \tilde{g}_\alpha\left(\frac{s}{L^\nu}\right). \tag{4}$$

where $\tilde{g}_\alpha(x) = x^{\frac{\beta}{\nu}} g_\alpha(x)$ and thus $\tau_\alpha = \frac{\beta}{\nu}$.
For larger system sizes the finite size dependence breaks down because the boundary governing this behaviour becomes increasingly negligible.

One is of course interested how the critical behaviour changes when disorder is introduced. A more general understanding of the effect is still lacking. Renormalization group calculations indicate that the critical behaviour is destroyed in the case of a random drive, i.e. when the driving is of the form

$$E_i \rightarrow E_i + \delta E \tag{5}$$

for a randomly chosen site i in each time step.
Let us finish this section with some remarks on the physical relevance of the models to real physical systems.

2.4 Physical Relevance of the Models

A number of experiments have been done on real sandpiles. Only small avalanches (up to about 50 grains) have been found to exhibit power law behaviour in their distributions. Real sandgrains tend to roll and thus accumulate kinetic energy, making inertia effects relevant. However, experiments on rice grain show behaviour more consistent with SOC. We expect this because rice grains have a smaller mass density. Moreover, they allow a larger number of metastable states of the system because of their elongated shape: they hardly roll. This is a better realization of the threshold dynamics of SOC systems. The forest fire model is closely related to excitable media which compromise phenomena such as spreading of diseases, oscillating chemical reactions or propagation of electrical activity in neurons. These systems essentially have 3 states: quiescent ("tree"), excited ("burning tree") and refractory ("empty site"). One can expect a SOC state in these systems for the appropriate range of parameter values, i.e., if excitation occues rarely and if excitation spreads much faster than the system recovers from the refractory state.

Recent investigations have revealed that the temporal record of the incidence of measles on some islands are very similar to the temporal variation in the number of fires predicted by the FFM.
The well-known empirical Gutenberg-Richter law states that the probability distribution of earthquakes follows a power law: $P(E) \sim E^{-B}$, where E is the energy released in an earthquake and B is an exponent that varies with the geographical location. Figure (4) shows the number of earthquakes per year in the New Madrid zone during the period 1974-83 as a function of the released energy.
Although the range of observed exponents B in real earthquakes lies within

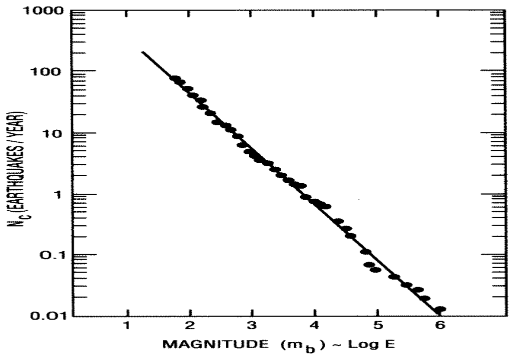

Figure 4: Example of an earthquake distribution.

the range of exponents τ_α in the OFC model the physical relevance of the model became questionable when it was found that modification of the open boundary conditions in a realistic way destroyed the SOC behaviour.

3 Analytical Approaches

3.1 Matrix Formalism For The Sandpile Model

For the critical height sandpile, where the state of a site depends only on the value of the dynamical variable at this site and not on its neighbours (in contrast to the critical slope model), several exact results have been obtained through a matrix formalism that exploits the Abelian character of the dynamics. Abelian means here that the order in which the perturbations to the model are applied (the driving) is irrelevant. It should not matter if we first add a particle to site A followed by an addition of a particle to site B or reverse the order. In this formalism the toppling rule is

$$h_i > h_{ic} \quad \Rightarrow \quad z_j \to z_j - \Delta_{ij} \qquad (j = 1, 2, ..., N) \tag{6}$$

where the elements Δ-matrix fulfill certain conditions according to the sandpile dynamics. One can calculate the two point correlation function which in the continuum approximation scales as $G(r) \sim r^{2-d}$. The height-height correlations $< z_i z_j > - < z_i >^2$ decay as r^{-2d}. Finally, using the Δ-formalism, the notion of inverse avalanches and the relation to spanning trees, Priezzhev, Ktitarev and Ivashkovich (1996) have calculated the probability distribution for the avalanche sizes $P(s) \sim s^{-\tau}$. The result is $\tau = \frac{5}{4}$ and is in excellent agreement with the dynamically driven renormalization group calculation which yielded $\tau = 1.253$.

3.2 Mean Field Approach

We will now outline a mean field treatment which includes dissipation explicitly. It turns out that the driving rate and the dissipation rate are control parameters that have to be tuned in order for the system to reach the critical state.

Consider stochastic SOC models (such as the sandpile model) in terms of three states: critical (site becomes active if energy is added), stable (site does not become active if energy is added) and active (relaxing site transfering energy). In a single site mean field approximation we can write

$$\frac{\partial}{\partial t}\rho_k = F_k(\rho_a, \rho_c, \rho_s) \qquad (k = a, c, s) \tag{7}$$

where ρ_a, ρ_c, ρ_s are the average densities of sites in the corresponding states. The explicit form of (7) can be derived from the master equation of the model. For our purpose it is sufficient to consider the expansion of F_k as a series of the average densities:

$$F_k = \sum_n f_k^n \rho_n + \sum_{n,l} f_k^{n,l} \rho_n \rho_l + \mathcal{O}(\rho_n^3) \tag{8}$$

The first order terms are the transition rates generated by the external driving field or by spontaneous transition. The second and higher order terms

characterize transitions due to interactions between different sites. Since the critical point is identified by $\rho_a = 0$ we can neglect second order terms in ρ_a. The effective parameters $f_k^n, f_k^{n,l}$ depend on the details of the model.
The considerations have been quite general so far. For the sake of clarity, let us take the example of the sandpile model. In this case we end up with the following equations:

$$\rho_a = h\rho_c + (g - \epsilon)\rho_c\rho_a \tag{9}$$
$$\rho_a = uh\rho_s + u(g - \epsilon)\rho_s\rho_a \tag{10}$$
$$\rho_a = 1 - \rho_s - \rho_c \tag{11}$$

where h is the driving rate (probability per unit time that a site will receive a grain of energy), g is an effective rate that depends on the geometry, ϵ is the average energy dissipated in each site (present also for fully conservative systems due to boundary dissipation) and u takes into account that only a fraction u of stable sites receiving an energy grain contributes to the $s \to c$ process. After some algebra and expanding $\rho_a(h)$ for small values of h one obtains

$$\rho_a(h) = \frac{uh}{1 + u - ug + u\epsilon}. \tag{12}$$

On the other hand, in the stationary state

$$J_{in} = hL^d = \epsilon\rho_a L^d = J_{out}, \tag{13}$$

where J_{in}, J_{out} are the average energy input/output fluxes. From Eq. (12) and Eq.(13) follows

$$\rho_a = \frac{h}{\epsilon} \qquad \rho_c = \frac{1}{g} + \mathcal{O}(h) \qquad \rho_s = \frac{g-1}{g} + \mathcal{O}(h). \tag{14}$$

The susceptibility is

$$\chi \equiv \frac{\partial\rho_a(h)}{\partial h} = \frac{1}{\epsilon}. \tag{15}$$

The one particle density of active sites is the order parameter and goes to zero at the critical point. We conclude that the model is critical in the double limit $h, \epsilon \to 0$, $\frac{h}{\epsilon} \to 0$. Thus the critical state arises due to the fine-tuning of the driving rate h and the dissipation rate ϵ. SOC appears as a special case of nonequilibrium critical phenomena, with the only peculiarity being that the critical parameters are zero. One may wonder how this fits together with the sandpile model. The answer is that the tuning conditions are implicitly enforced in the sandpile model: time scale separation (no driving during avalanches) and dissipaton only through the boundaries, which makes h and ϵ equal to zero in the thermodynamic limit.

3.3 Dynamically Driven Renormalization Group

A major problem in applying the renormalization group to nonequilibrium phenomena is that there is no Gibbs distribution like in ordinary (equilibrium) critical phenomena to assign a statistical weight to a certain configuration in the stationary state. The essential idea of the dynamically driven renormalization group (DDRG) is to combine a real space renormalization group (RSRG) with a dynamical steady-state condition which provides the nonequilibrium statistical weights. Although this nonequilibrium distribution neglects correlations finer scale correlations are included at the new scale because the new statistical weights are calculated from the steady-state condition with renormalized dynamical parameters. This scheme can be improved by considering increasingly better approximations in both the dynamical renormalization and the stationary distribution.
Consider a model defined on a d-dimensional lattice. To each site i of the lattice is associated a variable σ_i that can assume q different values ($\sigma_i = 1, 2, 3, ..., q$). A dynamics characterized by a set of parameters $\vec{\mu} \equiv \{\mu\}$ acts on these variables and defines the temporal evolution of the model. In the stationary state

$$\frac{\partial}{\partial t}\{< \rho_k >\} = \mathcal{S}_\mu(\{< \rho_k >\}) = 0, \tag{16}$$

where $\{< \rho_k >\}$ is the set of time independent average densities of sites in a particular state $\sigma_i = k \quad (k = 1, 2, .., q)$ and $\mathcal{S}_\mu$ is an operator describing the evolution of the system as a function of the dynamical parameters μ (we mean the whole set $\vec{\mu}$ when we write μ). We can write the probability that the system has a configuration σ at time t as

$$P(\sigma, t) = \sum_{\{\sigma^0\}} < \sigma|T(\mu)|\sigma^0 > P(\sigma^0, 0), \tag{17}$$

where $< \sigma|T(\mu)|\sigma^0 >$ is the transition probability from the configuration $\sigma^0 \equiv \{\sigma_i^0\}$ to the configuration $\sigma \equiv \{\sigma_i\}$ in a unit time step t and T is the discrete time evolution operator depending on the dynamical parameters μ. We then coarse-grain the system by rescaling length and time according to $x \to bx$ and $t \to b^z t$, where z is the dynamical critical exponent.
The renormalization transformation is constructed through a renormalization operator $\mathcal{R}(S,\sigma)$ with $S \equiv \{S_i\}$ being the set of coarse-grained variables. This operator rescales the lengths of the system. Usually, $\mathcal{R}$ corresponds to a block transformation in which lattice sites are grouped together in a supersite that defines the renormalized variables S_i by means of a majority or spanning rule. With

$$\mathcal{R}P(\sigma,t) = P'(S,t') \tag{18}$$

and the time-discretized Eq.(17)

$$P(\sigma,N) = \sum_{\{\sigma^0\}} < \sigma|T^N(\mu)|\sigma^0 > P(\sigma^0,0) \tag{19}$$

we obtain

$$P(S,t') = \sum_{\{\sigma\}} \mathcal{R}(S,\sigma) \sum_{\{\sigma^0\}} < \sigma|T^{b^z}(\mu)|\sigma^0 > P(\sigma^0,0) \tag{20}$$

where we have included $t' = b^z t$. By using

$$\sum_{\{S^0\}} \mathcal{R}(S^0,\sigma^0) = 1 \tag{21}$$

$$P'(S^0,0) = \sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0) P(\sigma^0,0) \tag{22}$$

this can be written as

$$P'(S,t') = \sum_{\{S^0\}} \left(\frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma) < \sigma|T^{b^z}(\mu)|\sigma^0 > P(\sigma^0,0)}{\sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0)P(\sigma^0,0)} \right) P'(S^0,0) \tag{23}$$

or

$$P'(S,t') = \sum_{\{S^0\}} < S|T'(\mu)|S^0 > P'(S^0,0) \tag{24}$$

which identifies the renormalized dynamical operator T'.
Applying this scheme to systems with a stationary distribution $P(\sigma, t \to \infty) = W(\sigma)$, we can study the renormalization of the dynamics for small deviations from the stationary state

$$P(\sigma,t) = W(\sigma) + \epsilon h(\sigma,t). \tag{25}$$

In the lowest order we take only into account the stationary distribution and obtain from Eq.(23) and (25)

$$< S|T'(\mu)|S^0 >= \frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma) < \sigma|T^{b^z}(\mu)|\sigma^0 > W(\sigma^0)}{\sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0) W(\sigma^0)} \tag{26}$$

Thus we see that the new dynamical operator T' is defined as the sum of the statistical weights of all the paths selected by the operator T^{b^z} that lead from any starting configuration that renormalizes in $\{S_i^0\}$ to any final configuration that renormalizes in $\{S_i\}$. Each path has a additional weight given by the relative probability of the stationary configuration $\{\sigma_i^0\}$, $W(\sigma^0)$.
The explicit form of the operator T^{b^z} depends on the details of the model and is rather tedious to obtain. The time scaling factor b^z is defined as the average number of steps that are necessary to preserve the form of the operator T, i.e., $T'(\mu) = T(\mu')$. From this condition we obtain the rescaled parameter set $\mu' = f(\mu)$ and it implies that the renormalized single time distribution $P'(S,t')$ has the same functional form as the original distribution $P(\sigma,t)$. Moreover, b^z is a function of the parameter set μ, $b^z = g(\mu)$, from which we can calculate the dynamical critical exponent z.
We still need to know how to calculate the steady-state distribution $W(\sigma)$. As an approximation we neglect correlations and take

$$W(\sigma) \approx \prod_i < \rho_{\sigma_i} > \tag{27}$$

as a mean field approximation, where $< \rho_k >$ is the average density of sites in the k-state. The densities $< \rho_k >$ can be obtained from the stationarity condition $\mathcal{S}_\mu(\{<\rho_k>\})$ (see the subsection "Mean Field Approach"). Inserting (27) into (26) we get the set of renormalization equations

$$< S\,|T'(\mu)|S^0 >= \frac{\sum_{\{\sigma^0\}}\sum_{\{\sigma\}}\mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma) < \sigma|T^{b^z}(\mu)|\sigma^0 > \prod_i < \rho_{\sigma_i^0} >}{\sum_{\{\sigma^0\}}\mathcal{R}(S^0,\sigma^0)\prod_i < \rho_{\sigma_i^0} >} \tag{28}$$

$$\mathcal{S}_\mu\,(\{<\rho_k>\}) = 0. \tag{29}$$

Eq.(29) is the dynamical steady-state condition that allows the evaluation of the approximate stationary distribution at each coarse-graining scale. We call it the *driving* condition since it drives the RG equations acting as a feedback on the scale transformation.
Rewriting these equations in the form of recursion relations and adding the equation for the rescaling of time we obtain:

$$\mu' = f(\mu, \{<\rho_k>\}) \tag{30}$$

$$t' = tg(\mu, \{<\rho_k>\}) \tag{31}$$

$$\mathcal{S}_\mu(\{<\rho_k>\}) = 0 \quad \Rightarrow \quad <\rho_k>= u_k(\mu). \tag{32}$$

From the fixed points $\mu^* = f(\mu^*)$ we can calculate the critical exponents z and τ. The dynamical critical exponent is according to (31)

$$z = \frac{\log g(\mu^*)}{\log b}. \tag{33}$$

One also obtains the average stationary properties of the critical state from

$$<\rho_k>^*= u_k(\mu^*). \tag{34}$$

Although in this form of the DDRG only the uncorrelated part of the steady-state distribution is taken into account the results are nontrivial because correlations in the system are considered in the dynamical renormalization of the operator T. The renormalized uncorrelated part of the stationary distribution is evaluated from the stationary condition with renormalized parameters, thus providing an effective treatment of correlations. However, it should be noted that the mean field treatment of the stationary distribution is limited by the fact that we are dealing with critical phenomena.

The renormalization scheme has been applied to the sandpile model. It has an attractive fixed point, therefore no fine tuning is necessary for the system to reach the critical state. The calculated critical exponents are in very good agreement with numerical simulations. Moreover it has been shown that the introduction of local dissipation destroys the critical behaviour. This is in agreement with our mean field result. The calculation for the forest fire model yields a repulsive fixed point, which corresponds to a critical state only reached by fine tuning. The essential difference between the sandpile model and the forest fire model is that the forest fire model cannot be studied in a subspace (i.e., without f and p) without destroying the model itself. This is possible in the case of the sandpile model in a subspace without the driving rate h. Including h would lead to repulsive fixed point and show that the system only reaches criticality for $h \to 0$.

4 Conclusions

Self-organized criticality was introduced as candidate for a theory of complexity. It was originally hoped that the concept would explain the abundant presence of fractal structures and the mysterious 1/f noise. However, this turned out to be not right. As we saw in the case of the earthquake model, it is difficult to simulate the behaviour of large systems. On the other hand, the Gutenberg- Richter law clearly shows a power law behaviour, indicating some kind of a critical state.

Even today a clear definition of self-organized criticality is still lacking. The recently developped analytical tools showed that it is closely related to a separation of time scales. Strictly speaking this means that self-organized criticality is in fact not self-organized because of the presence of relevant parameters.

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