Bose-Einstein Condensation in Confined Atomic Gases

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Abstract

Bose-Einstein Condensation is a phenomenon that occurs in boson systems where a finite fraction of the number of particles in the system occupy the single-particle ground state of the system. This phenomenon follows from the nature of Bose statistics and, therefore, behaves differently from a system of distinguishable particle.

In this term paper I will give a review of Bose statistics to prepare for the study of BEC. Then I will derive the criterion at which the phase transition to the BEC phase occurs. I will try to give some intuitive explanation why such a phenomenon has to occur in boson systems. I will also discuss some properties of the boson system that follow from the mathematical development of the problem, and I will give some physical explanations for these properties.

In the end I will discuss some of the experiments that have been carried out to produce and study BEC. I will also discuss a prospective application of the phenomenon.

1 Review of Bose Statistics

The main characteristic feature of Bose Statistics is that there is no limit on the number of pasrticles that can occupy the same single-particle quantum state. Another feature of the Statistics of identical particles is that the quantum state is identified by the number of particles in each single-particle state. Interchanging two identical bosons gives back the same state. The consequences of these two features can be seen clearly when one calculates the partition function for a system of identical bosons in an infinite potential well in the grand canonical ensemble.

$$Z(z,V,T) = \sum_{N=0}^{\infty} z^N Q_N(V,T) \tag{1}$$

where z is the fugacity and $Q_N(V,T)$ is the partition function in the canonical ensemble with N particles.

This partition function can be evaluated by considering the partition function for each energy level separately and then taking the product of the partition functions.

$$\begin{aligned} Z(z,V,T) &= \prod_{\mathbf{p}}[\textstyle\sum_{n=0}^{\infty}(ze^{-\beta\epsilon_{\mathbf{p}}})^n] \\ &= \prod_{\mathbf{p}} \frac{1}{1-ze^{-\beta\epsilon_{\mathbf{p}}}} \end{aligned} \tag{2}$$

This partition function gives the equation of state:

$$\begin{aligned} \frac{PV}{kT} &= \log Z(z,V,T) \\ &= -\textstyle\sum_{\mathbf{p}} \log(1-ze^{-\beta\epsilon_{\mathbf{p}}}) \end{aligned} \tag{3}$$

And with the condition that the number of particles in the sustem is given by N one finds the equation:

$$\begin{aligned} N &= z\frac{\partial}{\partial z} \log Z(z,V,T) \\ &= \textstyle\sum_{\mathbf{p}} \frac{ze^{-\beta\epsilon_{\mathbf{p}}}}{1-ze^{-\beta\epsilon_{\mathbf{p}}}} \end{aligned} \tag{4}$$

In the thermo dynamic limit the sums are replaced by integrals and the two equations describing the system are:

$$\frac{P}{kT} = -\frac{4\pi}{h^3}\int_0^{\infty} dp p^2 \log(1-ze^{-\beta p^2/2m}) - \frac{1}{V}\log(1-z) \tag{5}$$

$$\frac{1}{v} = \frac{4\pi}{h^3}\int_0^{\infty} dp p^2 \frac{1}{z^{-1}e^{\beta p^2/2m}-1} + \frac{1}{V}\frac{z}{1-z} \tag{6}$$

where $v = V/N$.

Or with the use of some special functions one can write this result as:

$$\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z) - \frac{1}{V} \log(1-z) \tag{7}$$

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V} \frac{z}{1-z} \tag{8}$$

where:

$$\lambda = \sqrt{2\pi\hbar^2/mkT} \tag{9}$$

$$g_j(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^j} \tag{10}$$

The quantity $z/(1-z)$ is the occupation number of the single-particle state with $\mathbf{p}$=0. It will be important below T_c when there is a finite fraction of the particles in this single-particle state.

At this point One can start looking for the criteria for a phase transition in this system.

2 How Bose-Einstein Condensation Occurs

Having reached at the equations describing a Bose system one can start considering the phase transition that occurs. This can be done simply by trying to find z in terms of the other parameters using equation (8). It can be written in the form:

$$\lambda^3 \frac{\langle n_0 \rangle}{V} = \frac{\lambda^3}{v} - g_{3/2}(z) \tag{11}$$

This can not be done analytically, because one cannot handle the function $g_{3/2}(z)$ easily. However, it can be solved graphically (At least one can understand why the phase transition occurs). Figure (1) shows the function $g_{3/2}(z)$. Figure (2) shows how the equation can be solved graphically:

It should be noticed that the curve due to the occupation of the ground state gets narrower, the larger V gets, and looks like a delta function in the thermodynamic limit. If λ^3/v is smaller than $g_{3/2}(1)$, the equation can be solved and z will take some value between 0 and 1. However, if λ^3/v is greater than $g_{3/2}(1)$, z will take a value that is almost equal to 1 and that actually converges to 1 in the thermodynamic limit. One can also look at it this way: Moving up the vertical axis z changes between 0 and 1 until one reaches the point $\lambda^3/v = g_{3/2}(1)$, after which z remains constant at the value 1, no matter how high one goes up that axis. Therefore the criterion that can be used to determine the critical temperature for BEC can be written as follows:

$$\frac{\lambda^3}{v} = g_{3/2}(1) = 2.612 \tag{12}$$

or

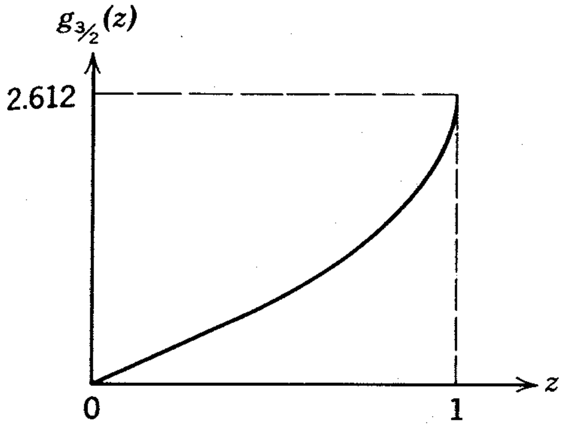

Figure 1: The function $g_{3/2}(z)$

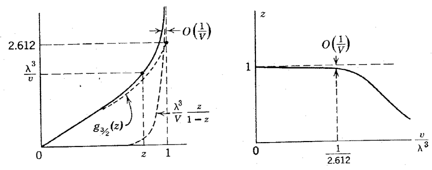

Figure 2: Fraction of particles in the ground state

$$kT_c = \frac{2\pi\hbar^2/m}{[v g_{3/2}(1)]^{2/3}} \tag{13}$$

Or if the temperature is held constant and the particle density is varied, the critical volume would be:

$$v_c = \frac{[2\pi\hbar^2/mkT]^{3/2}}{g_{3/2}(1)} \tag{14}$$

Below T_c, z hardly changes, while the fraction of of particles in the ground state rises steeply to reach 1 at absolute zero temperature. The fraction of particles in the ground state can be written as a function of T and T_c :

$$\frac{\langle n_0 \rangle}{N} = \begin{cases} 0 & \text{if } T > T_c \\ 1 - (\frac{T}{T_c})^{3/2} & \text{if } T < T_c \end{cases} \tag{15}$$

or

$$\frac{\langle n_0 \rangle}{N} = \begin{cases} 0 & \text{if } v > v_c \\ 1 - (\frac{v}{v_c}) & \text{if } v < v_c \end{cases} \tag{16}$$

Figure (3) shows this fraction as a function of temperature:

One can get some intuitive feeling of why this phase transition occurs by taking a system at constant temperature while increasing the number of particles. An

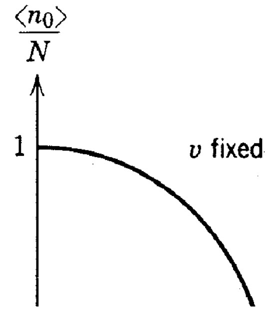

Figure 3: Average occupation number of the ground state

important point is that the energy of a many-particle state relative to the ground-state energy is the sum of the energies of all particles. This means that the energy of having a certain fraction of the particles in one of the excited states does not depend only on the energy of that state, but also on the number of particles in the system. One might think that it is the same case with distinguishable particles, and here one sees the effect of having a system of identical particles. With distinguishable particles the degeneracy (due to exchange of particles) of the excited states cancels the effect of high energies. That is why the fraction of particles in excited states is independant of the number of particles in such systems.

All of this discussion did not involve interactions between particles. If one wishes to include these interactions, there will be some small corrections if the interactions are weak. If the interactions are strong, the Hamiltonian that I started with will not be suitable for the problem, and the problem has to be solved in a different way. It is usually impossible to find exact results if interactions are included. The critical temperature, order parameter, P-V diagram etc. will be modified as a result of interactions.

3 Properties of a Bose-Einstein Condensate

The fugacity and fraction of particles in the condensate have been calculated and figures of them were shown in the previous section. One can go on and study a number of other properties of the condensate. For example one can go back to the equation of state (7) and draw a P-V diagram for the system:

For an isotherm at high specific volumes the boson gas behaves like a classical Boltzmann gas. This is natural because in that limit quantum corrections are negligible and it does not matter what statistics the system follows. As one gets close to the transition line quantum corrections start to affect the shape of the isotherm and it no longer looks like that of a classical gas. The most interesting part, however, is that below the transition specific volume. In that region the pressure remains constant until V reaches zero. In the thermodynamic limit the last term in equation (7) is negligible, and therefore the pressure is a function of the temperature only.

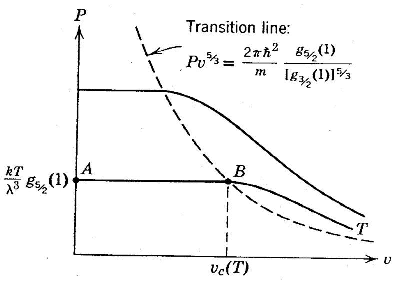

Figure 4: Isotherms of the ideal Bose gas

From that equation it can be seen that below v_c:

$$P = (\frac{m}{2\pi\hbar^2})^{3/2} g_{5/2}(1)(kT)^{5/2} \tag{17}$$

This occurs because the **p**=0 state does not depend on the size of the system and, therfore, can be said to have zero specific volume. This means that the condensate has no specific volume (or can be compressed to that volume without causing any "trouble". The noncondenced particles have some finite specific volume. But what happens is that if the system is compressed after forming the condensate, the effect of the compression will be to drive more particles into the condensate until eventually all the particles are in the condensate and the size of the system vanishes (however, that does not happen near $v = 0$because of other reasons that were not taken into account here).

Two other properties that are usually studied are the internal energy per particle and specific heat of the system:

$$\frac{U}{N} = \frac{3}{2}Pv = \begin{cases} \frac{3}{2}\frac{kTv}{\lambda^3} g_{5/2}(z) & \text{if } T > T_c \\ \\ \frac{3}{2}\frac{kTv}{\lambda^3} g_{5/2}(1) & \text{if } T < T_c \end{cases} \tag{18}$$

$$\frac{C_V}{Nk} = \begin{cases} \frac{15}{4}\frac{v}{\lambda^3} g_{5/2}(z) - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} & \text{if } T > T_c \\ \\ \frac{15}{4}\frac{v}{\lambda^3} g_{5/2}(1) & \text{if } T < T_c \end{cases} \tag{19}$$

Figure (5) shows the specific heat of the system as a function of temperature:

As the temperature decreases above T_c the specific heat increases as expected for an ideal gas. At T_c there is a cusp in the specific heat and below T_c the specific heat decreases until it vanishes at zero temperature. The reason behind that is that as particles condense, they do not have any specific heat any more. So the lower one goes in temperature, the more particles condense, and the smaller the "effective" system gets. At zero temperature all the particles are in the condensate and there is no "effective" system that can contribute to the specific heat.

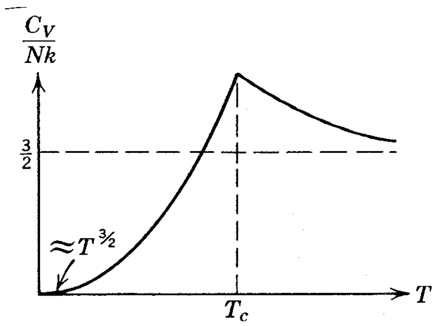

Figure 5: Specific heat of the ideal Bose gas

4 Recent Experiments Studying Bose-Einstein Condensation

Bose-Einstein condensation was observed experimentally for the first time in 1995 by a group in Boulder, Colorado. A cloud of atoms is trapped in an inhomogeneous magnetic field. The atoms whose magnetic moments are pointing opposite to the direction of the magnetic field escape and only those with magnetic moments parallel to the field get trapped at the centre of the field. A number of laser beams are then directed towards the cloud with a frequency slightly too low for stationary atoms to absorb and start moving. However, if an atom is moving opposite to the laser beam within a certain range of energies, it will absorb a photon and slow down. That happens because of the doppler shift that moving atoms experience. This process can produce a cloud of atoms at a temperature of the order of 1μK. Afterwards the cloud is cooled down further using evaporative cooling. The energy of the trap is designed to be the energy of atoms that are desired to be kept in the trap. Atoms with energies higher than that will be able to escape the trap, and only those atoms with low energies stay in the trap. This process takes the cloud to a temperature of the order of 100nK. At that moment it can be said that there is a BEC at the centre of the trap, but it is still difficult to observe directly. This condensate is then allowed to expand. The atoms with high energies expand quickly forming a large cloud and the atoms of the condensate hardly move forming a peak of high density at the centre of the cloud. After waiting a certain period of time a laser beam is directed towards the cloud and is diffracted from it forming a picture that represents the density of the cloud. This is how the density peak was observed at the centre of the cloud.

A number of experiments followed. BEC was produced at MIT and Rice with different isotopes, but almost the same idea.

Then they started to study the properties of condensates. In one experiment done at MIT the electromagnetic fields were turned off and the condensate was allowed to fall under the influence of gravity. The condensate fell as one cloud. Some people might consider that not very impressive.

In another, more interesting experiment done also at MIT a condensate was divided into two by shining a laser sheet at the middle of the magnetic trap. This results in having two condensates on the two sides of the laser sheet. Then both the magnetic and laser fields were turned off and the two condensates were allowed to expand and collide with each other. A picture was then taken of the configuration and an interference pattern of visibilty between 50-100% was observed. This experiment is an illustration of the fact that it is the wavefunctions of the condensates that are causing the interference pattern and that we really have a BEC.

In december 1997 the MIT group were able to trap the condensate in a laser field without the use of a magnetic trap. This eliminated the problem that only one spin direction can be trapped, and allowed the magnetic field to be an adjustable parameter of the experiment, rather than restricting it to be the confining potential.

5 Possible Applications of Bose-Einstein Condensates

There is no clear picture of what the applications of BEC will be, as far as I know. Inspired by the analogy between BEC and coherent light an idea that would usually come to mind is the atom laser. If the properties of BEC are understood well and it is possible to manipulate the condensate, pushing it with the desired speed in the desired direction, one can have an atom laser. It will have the same properties of the laser being a large number of coherent particles. However, the particles in the atom laser will also have a finite mass. This means they will interact with matter much stronger than photons, resulting in shorter penetration lengths and more "destructive" powers. Personally, I do not know what the applications of that would be, but there will certainly be some applications requiring atom lasers.

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