Non-Equilibrium Scaling in Stellar Phenomena

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Abstract

Many non-equilibrium systems show power law behavior in some way, whether it is noise, durations of avalanche events, or energy spectra. Inspired by the success of scaling laws in continuous phase transitions, self-organized criticality postulates that this power law behavior results from proximity to a kind of critical point. Initially appealing for its simplicity, experiments have brought its plausibility into question. Renormalization group methods inspired by those developed for continuous phase transitions have emerged as more successful alternative. This paper compares these two approaches to non-equilibrium scaling by considering their application to astrophysical phenomena, specifically solar flares and dimming events in the star KIC 8462852.

1 Introduction

A feature that seems to be ubiquitous in non-equilibrium phenomena is power law behavior, or "non-equilibrium scaling" (NES). An example is "$1/f$ noise" (sometimes called "pink noise" or "flicker noise"), where equal noise power is found in all frequency decades. This has been found to occur in many natural systems, including river flows and tides, resistivity in condensed matter systems, heart and neural activity in humans, and even electromagnetic bursts in astronomical bodies such as stars and quasars [2]. Another prominent example is avalanches or crackling, where the system undergoes sudden jumps or bursts in response to an external agent. They were originally observed in ferromagnets: if an external magnetic field is applied to a sample then slowly varied, the sample's magnetization will discretely change in events termed "avalanches," where a domain of spins in the sample will abruptly align with the external field. This is called the Barkhausen effect. The size, or number of spins in, such flips is found to obey a power law distribution [6].

Several models have been proposed to explain NES. The first was self-organized criticality (SOC). In SOC, the system will organize itself into an equilibrium point such that the dynamics resemble those near a critical point in continuous phase transitions. The power law behavior thus follows from this similarity. The major difference is that this critical point is reached without tuning any experimental parameters: the system simply reaches the point as a part of its natural evolution [3]. However, experiments have had considerable difficulty demonstrating SOC: they have been unable to reproduce the power law behavior that is expected. Moreover, SOC simulations fail to offer physical insight into an underlying physical mechanism. It is currently seen as an oversimplification of true avalanche dynamics [4].

Inspired by SOC and the success of the renormalization group (RG) in predicting scaling behavior in continuous phase transitions, an RG theory of avalanches has been developed in which the observed power laws power laws follow from RG flows. Avalanches are seen as a phase boundary between a complete response from the whole system and random, intermittent responses from small parts of the system. It has successfully predicted scaling in the ferromagnetic noise described above. Since RG relies on an underlying physical model and accurately predicts scaling in avalanches, it is currently is preferred to SOC [6].

A more elusive example of non-equilibrium power laws lies in solar flares, where brief spikes in power are seen in the sun's spectrum. It is generally agreed that these flares result from charged particles interacting with solar plasmas, but the precise emission mechanism is still unknown. Particularly strong power laws have been observed in x-ray and gamma ray bursts when considering the peak intensity, flare duration, and total intensity. The prevalence of power laws made solar flares and early application of SOC [1]. More recently, such power law behavior has been observed in dimming events of the star KIC 8462852 (more commonly known as Tabby's Star), but the source of these events remains a mystery. The avalanche RG approach has recently been applied to this star in an attempt to explain the observed scaling [7].

This paper uses the attempted modeling of the aforementioned stellar phenomena to compare the SOC and the RG approaches. We begin with an overview of SOC and its experimental shortcomings, then explain how it has been used in attempts to model solar flares. We then summarize RG avalanche theory and then show that the mean field theory (MFT) exponents it predicts are consistent with those found in Tabby's Star dimming events.

2 Self-Organized Criticality

2.1 Description of SOC

In 1987, Bak, Tang, and Wiesenfeld (BTW) introduced self-organized criticality (SOC) as an explanation of the "$1/f$ noise" problem and the origin of fractal behavior in physical systems. They determine which non-equilibrium systems exhibit power law behavior by analyzing how the system responds to small perturbations. The original argument is quite simple and general, so we repeat it here.

Consider an interacting system, say coupled, damped, torsional pendula in a gravitational field. Take the system far from equilibrium by applying random disturbances, then allow the system to relax. The system may not return to its true ground state, where all the pendula have no tension; rather, it will probably go to a state where the pendula are still far from their individual equilibria but the net torque on each is still zero. BTW call such a state "barely stable," as a perturbation will disrupt the system. Imagine moving one pendulum to a new position in a one dimensional chain: the disturbance will propagate down the chain forcing each pendulum to a new position until the torques are equal to their original values and the system is once a gain minimally stable. Now imagine a two or higher dimensional grid of such pendula. If one pendulum is disturbed, the dynamics are nontrivial since the interactions are no longer "one-to-one" as in the one dimensional case. BTW claim that the system will continue to evolve until noise in the system cannot propagate at infinite distances. At this point, there is no length scale in the problem and we expect asymptotic scaling behavior, just as there is near a critical point in equilibrium statistical mechanics. Hence, the system has self-organized around some "critical-like point." It is important to emphasize that such a point is not a true critical point: the only similarity is the scaling behavior.

To illustrate their ideas, BTW numerically simulate a "sand pile:" a mound of small particles in a gravitational field will collapse on itself, or undergo and avalanche, if the local slope exceeds a critical value, like a pile of sand (if you don't believe me, go out to your garden and try it). Their original simulation was in the style of cellular automota. For simplicity we present the evolution scheme in two dimensions. A discrete variable $z(x,y)$ is defined to represent the local slope of the pile at (x,y). The grid is initialized to random values of z. If a particular value exceeds a critical value K, it and its neighbors are evolved as follows:

$$z(x,y) \rightarrow z(x,y) - 4 \tag{1}$$

$$z(x \pm 1, y) \rightarrow z(x \pm 1, y) - 1 \tag{2}$$

$$z(x, y \pm 1) \rightarrow z(x, y \pm 1) - 1 \tag{3}$$

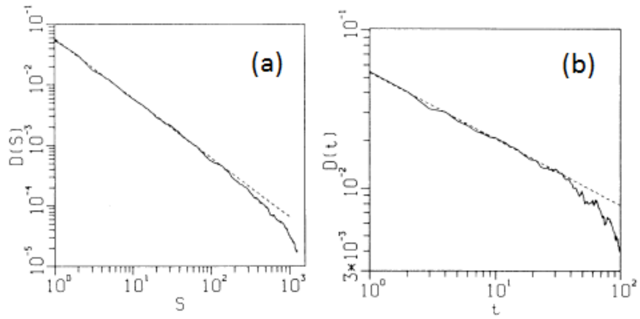

Figure 1: Power law relationships for sand pile simulations in two dimensions. The distribution of event sizes is shown in graph (a) and the distribution of event durations is shown in graph (b). The power laws mimic pink noise. (Taken from [3])

The results of this two dimensional simulation are presented in Figure 1. Graph (a) shows the distribution of event sizes s (that is, how many events affecting s particles took place) and graph (b) shows the distribution of event durations t. We see that an inverse power law emerges that is reminiscent of the "$1/f$ noise" distribution [3].

SOC is a very appealing theory since it is ostensibly quite general, requires no experimental tuning to enact, and would explain elusive physical phenomena such as NES.

2.2 Experimental Difficulties with SOC

Almost immediately after SOC was proposed, there were experimental efforts to test its validity by studying avalanches in real piles of granular material. In particular, Jaeger, Liu, and Nagel (JLN) performed an experiment in which small particles of different materials were dropped in different manners into different containers. The space between the resulting avalanches and their durations were documented and organized into histograms. The results were not power laws spanning several decades, but instead peaked distributions, as shown in Figure 2(a). JLN attribute this to the existence of a hysteresis slope: avalanches do not occur at the observed relaxation slope as in BTW's model, but when the slope exceeds the relaxation slope by a certain amount that is material dependent. That is, avalanches begin when the local slope is $s+\delta$ and the system comes to rest when the slope is s.

JLN further investigate whether SOC is achieved if the hysteresis slope is removed: mechanical vibrations are applied to the particle's container at different frequencies while the (cylindrical) container was rotated at a constant rate, ensuring the system can never come to rest and effectively setting $\delta \to 0$. The power spectrum as a function of the vibration frequency is shown in Figure 2(b). Different systems were characterized by different steady state angles θ_{ss}, the angle (related to the slope) of the particles when the vibrations were turned off, but the container still rotated at a constant rate. A power law is seen over a very narrow range, but not nearly as wide as predicted by SOC.

Finally, the container rotation was set to zero and the vibration frequency was varied while a power spectrum for the avalanches was recorded. A logarithmic dependence was

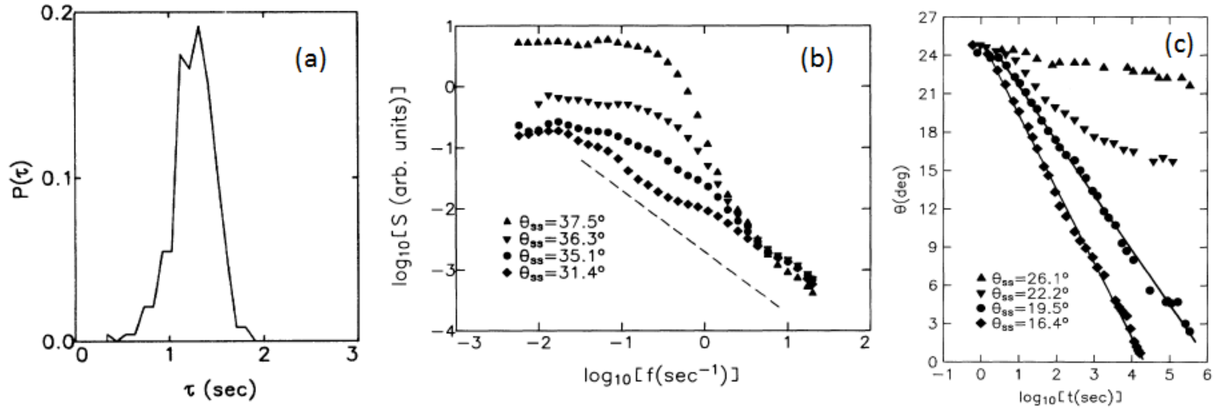

Figure 2: Experimental results of avalanches is systems of small particles. (a) is a distribution of avalanche durations when particles are added to the system; note that it is peaked, not a power law. (b) is the power spectrum of avalanches in rotating container with mechanical vibrations. (c) is a power spectrum in a stationary container with mechanical vibrations. A power law does not emerge in any of these cases as SOC predicts. (Taken from [4])

observed in some cases instead of the expected power law, as shown in Figure 2(c) [4].

This and other experimental efforts demonstrate that SOC is, at best, an oversimplification of true avalanche dynamics and one should be cautious in applying it.

3 Solar Flares

3.1 Observed Power Laws

One of the first applications of SOC was an attempted explanation of solar flares, where power laws similar to the ones described for sand piles are well documented. Power law behavior is especially apparent for what are called "hard x-rays," in which nonthermal particles are accelerated and emit bremsstrahlung radiation in the x-ray regime. Three power law exponents have been identified for solar flare data: peak intensity, total energy flux, and event duration. These power laws represent the number distributions of the specified quantity. For example, the event duration distribution describes the number of events which lasted for a particular length of time. Hard x-ray data are summarized in Aschwanden's review article. The three exponents obtained from the results of three solar surveys, each spanning a decade between 1980-2010, are presented in Table 1. Graphs of the results of three surveys are presented in Figure 3 [1].

3.2 Attempted SOC Explanation

All attempted SOC models for solar flares seem to rely on cellular automaton simulations in the same vein as BTW's original work. In particular, the simulation by Lu, Hamilton,

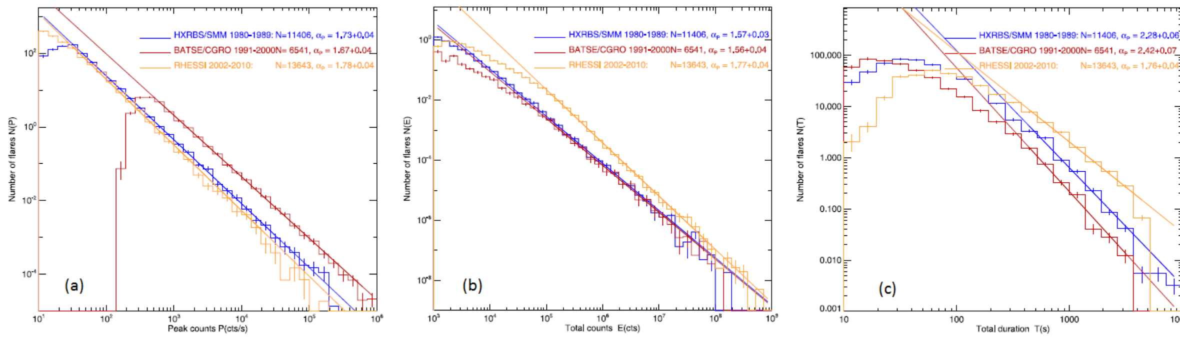

Figure 3: Data for solar flares covering the years 1980-2010. Each color represents a different survey. (a) shows the peak intensity (or peak count rate) distribution. (b) shows the total energy flux (or total number of counts). (c) shows the flare duration. (Taken from [1])

α_P	1.73 ± 0.07
α_E	1.62 ± 0.12
α_T	1.99 ± 0.35

Table 1: Observed solar flare exponents obtained from three surveys. α_P represents the exponent for peak activity. α_E is for event size. α_T is for event duration. (Taken from [1])

McTiernan, and Bromund (LHMB) successfully reproduced the observed exponents in solar flares. A vector field $\mathbf{F}_i$ is assigned to each point on a lattice (LHMB identify this field with an electromagnetic vector potential). A local "slope" is defined by

$$d\mathbf{F}_i = \mathbf{F}_i - \sum_j w_j \mathbf{F}_{i+j} \tag{4}$$

where the sum over j denotes the sites interacting with i. If $|d\mathbf{F}_i|$ exceeds a critical value F_c, the site and its neighbors are adjusted by

$$\mathbf{F}_i \to \mathbf{F}_i + \mathbf{f}_0 \tag{5}$$

$$\mathbf{F}_{i+j} \to \mathbf{F}_{i+j} + \mathbf{f}_j \tag{6}$$

w_j and $\mathbf{f}_j$ are chosen such that the sums of $\boldsymbol{F}$ and $d\mathbf{F}$ are conserved over the whole lattice. The system is driven by adding a small perturbation $\delta\mathbf{F}$ to a randomly selected lattice point. If $|d\mathbf{F}_i|$ is pushed past F_c, site and its neighbors are evolved according to Equations 5 and 6. If a neighbor is pushed passed F_c, that site is evolved. This process continues until all lattice sites are below the threshold. Such an evolution is identified with an avalanche. This process is repeated many times and the size (total number of critical sites in an event), duration (number of iterations until the system relaxes), and peak activity (maximum number of critical sites in a particular event) for each event are recorded. Power law behavior over wide ranges was found for each quantity. In addition, the scaling exponents were found to be largely insensitive to the number of lattice points. We quote the values for a 50×50×50 grid, being the largest grid size, in Table 2. They agree reasonably well with the exponents shown in Figure 3 [5].

α_P	1.86
α_E	1.51
α_T	1.88

Table 2: Power law exponents for LHMB's [5] simulation on a 50×50×50 grid. Compare with the values in Table 1.

Although it is tempting to believe that these results hold promise for a physical explanation, it is also important to keep in mind the experimental shortcomings of SOC-type models discussed in Section 2.2. Also note that the model neither offers insight into the physical origin of the avalanches nor furnishes a mechanism for the release of radiation seen in solar flares. Because SOC models do not provide insight into important aspects of the system such as these, SOC has not gained acceptance as a plausible physical model.

4 Avalanches in Renormalization Group Formalism

We now turn to a more widely accepted explanation for non-equilibrium scaling. Inspired by the success of RG methods in continuous phase transitions, such methods were applied to explain avalanches. It has successfully explained aspects of non-equilibrium behavior such as scale invariance and avalanche power laws in ferromagnet domain formation. Furthermore, it provides physical insights such as a fundamental mechanism and experimental tuning parameters. We summarize the main results in this section. Rather than go through a detailed mathematical treatment, we give a qualitative discussion following Sethna, Dahmen, and Myers.

There are several ways a system can respond to an external force. For example, when a twig is snapped, the entire system is correlated and responds instantaneously. When popcorn is heated and popped, the system responds in small, uncorrelated bursts. However, if the correlations between system components are stronger than random popping and weaker than instantaneous snapping, large portions of the system will discretely respond as the external force is slowly applied in events called "avalanches" or "crackles" (yes, this was a build-up to Snap, Crackle, Pop). Prominent examples of avalanches include domain flipping in ferromagnets when an external magnetic field is applied and tectonic plates suddenly shifting in earthquakes.

Since the processes described above are time-dependent, they are not in equilibrium and traditional statistical mechanics cannot be applied. Such processes can still be mathematically modeled, though. As an example, consider a ferromagnetic domain i under the influence of an external magnetic field. The force on this domain can be represented

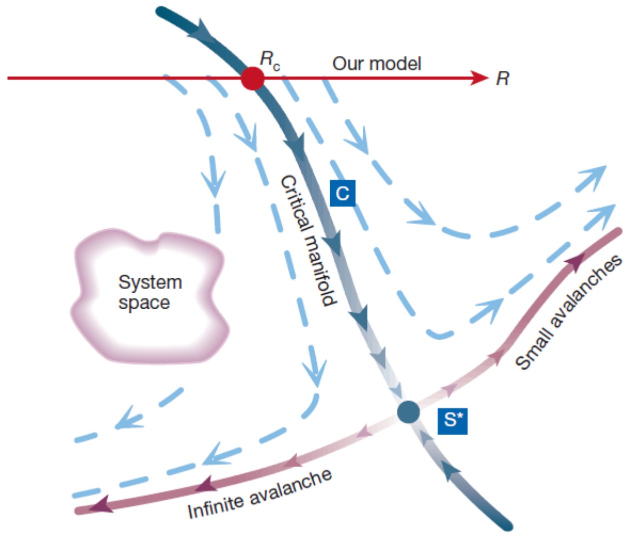

Figure 4: A projection of an RG flow in real space for an avalanche system. The critical manifold **C** flows to the fixed point S^*. The two unstable directions represent infinite avalanches and small avalanches, or snapping and popping respectively as discussed above. A particular model is represented by the red line; it interects the critical manifold at $R = R_c$. (Taken from [6])

as

$$F(i,t) = H(t) + \sum_{j \epsilon i} J_j S_j + h_i \tag{7}$$

where $H(t)$ is the time-dependent external field, J_j are the coupling constants between spins in the domain, and h_i is a parameter describing random factors such as the domain size and shape. h_i is selected from a normal distribution of domain sizes and shapes with standard deviation R.

Since we now have an interaction model with coupling constants (namely $H(t)$, J_j, and h_i), RG methods can be applied. A two dimensional projection of possible real-space results are shown in Figure 4. A fixed point S^* is found using a recursion relation, and relevant and irrelevant directions are identified. The fixed point represents a model which maps into itself and, hence, is truly scale invariant. This model represents the self-similarity that is commonly observed in avalanche systems. There exists a critical manifold **C** that flows into S^* dividing the space into two parts. The flow of the unstable direction gives these two parts meaning: models to the left of **C** describe one large snap, or an "infinite avalanche," as described above; models to the right of **C** describe smaller popping events as described above. If a model lies on **C**, it will flow into S^* and describe avalanches in the domains. The particular model described in Equation 7 is shown by the red line. It intersects **C** at a critical value of R labeled R_c. If R has this critical value the model flows into S^* and describes avalanche events. Thus, **C** represents a kind of phase transition [6].

We now list the critical exponents that this theory predicts. Let S be the size of an avalanche (in our ferromagnet example the number of spins in a domain), T be the temporal duration of an avalanche, $C(S)$ be the probability that an avalanche has a size greater than S, and $C(T)$ be the probability that an avalanche lasts longer than T. Then the following power laws hold near the critical point:

$$T \sim S^{\sigma\nu z} \tag{8}$$

$$C(S) \sim S^{-\tau+1} \tag{9}$$

$$C(T) \sim T^{-\alpha+1} \tag{10}$$

σ, ν, and z have separate definitions that are irrelevant to this discussion. Note that if (8) is substituted into (9) and compared with (10), the following relationship between the exponents emerges [6]:

$$\frac{\tau - 1}{\alpha - 1} = \sigma\nu z \tag{11}$$

If mean field theory techniques are applied to these types of models, then the following values are predicted for these exponents:

α	2
τ	3/2
$\sigma\nu z$	1/2

Table 3: MFT critical exponents for the RG approach (Taken from [7])

5 Tabby's Star

We turn our discussion to the observed properties of the star KIC 8462852, or "Tabby's Star." The spacecraft *Kepler* has taken data on time variations in the intensity of emitted light for approximately 150 000 stars. Intensity in a particular wavelength range is called luminous flux, or flux for short. The craft's original goal was to detect exoplanets: if periodic drops in a star's intensity are observed, astronomers take this as evidence of an orbiting satellite. Instead of periodic drops in intensity, drops were observed that seemed to have random duration and random spacing in the case of Tabby's Star.

Sheikh, Weaver, and Dahmen (SWD) argue that this is evidence of proximity to a non-equilibrium critical point, in analogy with avalanche systems. They define an avalanche to be an event where the intensity drops below the median intensity for all data taken. The duration of an event is defined to be the amount of time the intensity remains below the median. The size of an event is defined to be the time integral of the intensity over the event's duration, or the radiant energy flux as is typically measured in solar flares. In Figure 5, a few avalanche events are shown in the light curve data over several days. The size of the event is the area between the light curve and the median over the duration.

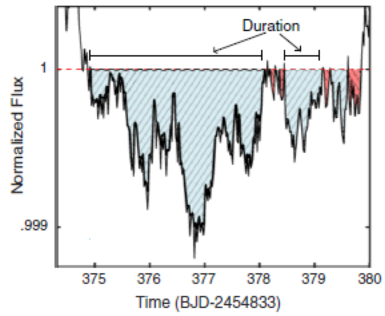

Figure 5: A few avalanches as seen in the light curve data for KIC 8462852. 1 on the vertical axis represents the median intensity; any dimming below this value is considered an avalanche. The duration is the total time spent below this point and the size is the area of the curve below this point, as colored in the plot. Different colors represent different avalanches. (Taken from [7])

In analyzing light curve data for this star, they are able to observe the power laws described in the previous section. The graphs are shown in Figure 6 and the observed exponents are tabulated in Table 4.

α	$2.09^{+0.07}_{-0.40}$
τ	$1.60^{+0.10}_{-0.13}$
$\sigma\nu z$	$0.67^{+0.04}_{-0.02}$
$\frac{\tau-1}{\alpha-1}$	$0.55^{+0.46}_{-0.14}$

Table 4: Observed Exponents for KIC 8462852. The observed values are close to MFT exponents but not in exact agreement with them. (Taken from [7])

The reported values are consistent with those of MFT, even if they are not in complete agreement. This would suggest that the star is near a critical point of some sort. A complete analysis would require a detailed RG model for some stellar process to be developed in which tuning parameters are manifest, and no such model exists to the author's knowledge. SWD point out that the scaling behavior of KIC 8462852 mimics three other stars known to be magnetically active, so the underlying mechanism is probably magnetic or magnetohydrodynamic in origin. Although the MFT analysis of Tabby's Star dimming events still does not provide an underlying physical explanation, the fact that the scaling exponents agree with observation demonstrates that there is an avalanche mechanism that can explain these events [7].

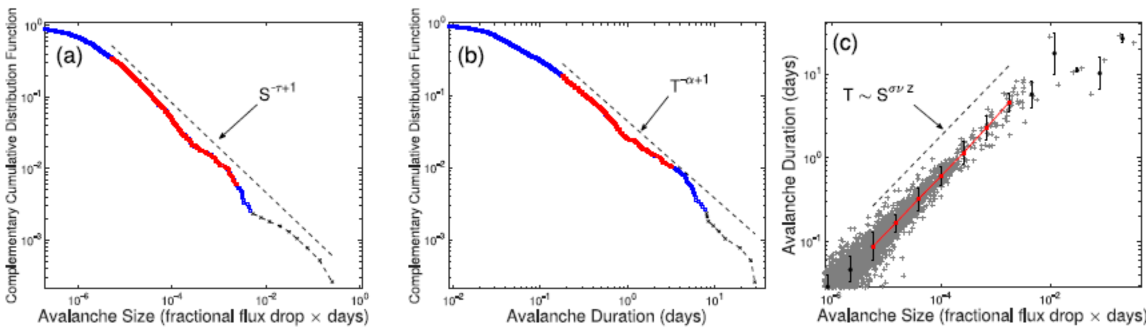

Figure 6: Power laws in KIC 8462852. Graph (a) demonstrates the $C(S)$ power law spanning approximately two decades. Graph (b) demonstrates the $C(T)$ power law spanning approximately one decade. The red parts of the curve indicate the scaling regime. Graph (c) demonstrates the T vs S power law; this law appears to only be valid for small events. (Taken from [7])

6 Conclusions

We have presented two explanations for power law behavior in non-equilibrium phenomena: self-organized criticality and renormalization group based avalanche phase transitions. SOC models are ostensibly appealing for their simplicity, but they fail to provide insight into the fundamental physics and often neglect key experimental parameters, as in the case of sand pile models. These shortcomings were seen in the attempted explanation of scaling in solar flares: the power laws were reproduced, but it was with a simulation that had no grounding in physical theory and offered no insights into how the avalanche events transpired. RG methods, on the other hand, use an underlying physical model to generate recursion relations and determine observable quantities such as scaling laws and phase boundaries. Although the underlying mechanism has not been determined for dimming events in Tabby's Star, the consistency with MFT exponents in avalanche RG strongly suggests that some avalanche model is key to the star's dynamics.

7 References

[1] M. J. Aschwanden. *The State of Self-organized Criticality of the Sun During the Last Three Solar Cycles. I. Observations.* Solar Phys. (2011) 274.

[2] M. J. Aschwanden, N.B. Crosby, M. Dimitropoulou, et al. *25 Years of Self-Organized Criticality: Solar and Astrophysics.* Space Sci. Rev. (2016) 198: 47.

[3] P. Bak, C. Tang, and K. Wiesenfeld. *Self-Organized Criticality: An Explanation of 1/f Noise.* Phys. Rev. Lett. (1987) 59: 4.

[4] H. M. Jaeger, C. Liu, and S. R. Nagel. *Relaxation at the Angle of Repose.* Phys.

Rev. Lett. (1989) 62: 1.

[5] E. T. Lu, R. J. Hamilton, J. M. McTiernan, K. R. Bromund. *Solar Flares and Avalanches in Driven Dissipative Systems.* Astrophys. Journal. (1993) 412: 841-852.

[6] J. P. Sethna, K. A. Dahmen, and C. R. Meyers. *Crackling Noise.* Nature. (2001) 410.

[7] M. A. Sheikh, R. L. Weaver, and K. A. Dahmen. Avalanche Statistics Identify Intrinsic Stellar Processes near Criticality in KIC 8462852. Phys. Rev. Lett. (2016) 117: 261101.