Holographic Wilsonian Renormalization Group

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Abstract

Strongly coupled systems are difficult to study because the perturbation of the systems does not work with strong couplings. However, the gauge/gravity duality and the ultraviolet-infrared connection enable to study the low energy regime of the gravity. The fundamentals of holographic Wilsonian renormalization is introduced with the example of free massive scalar fields. Finally, the crucial point of the holographic Wilsonian renormalization group and further questions will be discussed compared to the previous renormalization group methods.

1 Introduction

By its nature of strong couplings, the strongly coupled systems have not yet been well studied by perturbation. One approach to study the strongly coupled systems is looking at the low energy regime of the system by integrating out the high energy part. In the case of gravity, we can use the gauge/gravity duality and the UV-IR connection to set up the holographic Wilsonian renormalization Group. In this paper, I will review the basics of holographic Wilsonian renormalization group, which involves the gauge/gravity theory duality and the Wilsonian approach to the renormalization group[RG].

The gauge/gravity duality is the concept that the gravity in the bulk space has a correspondence with the gauge theory on the boundary of the bulk space. Anti de-Sitter/Conformal Field Theory[AdS/CFT] is a good example to study the duality; the physics in a geometrically AdS space is exactly same as the physics of the quantum field theory on the boundary space. In this paper, I will use the duality that the gravitational field in the bulk is asymptotically AdS, and the boundary field theory is $\mathcal{N}$-4 super Yang Mills theory. The duality enables to understand the gravity by studying the boundary field theory.

The Wilsonian RG transforms an action with coupling constants to the new action and the new coupling constants. The transformation integrates out the high energy microscopic degrees of freedom, and return the macroscopic degrees of freedom. The newly calculated action and the coupling constants by the scaling transform are different from the old ones, but the symmetries of the action are all preserved. Hence, one can study the physics up to desired scale by the repeating the transformation. In this paper, I will focus on the low energy physics by integrating out the high energy regime as well as setting the cut-off energy scale Λ_0.

In the holographic Wilsonian RG, the high energy part of the boundary theory($\Lambda > \Lambda_0$) is integrated out, and this corresponds to the integrated out bulk region of $z < \epsilon_0$. We will see that the radial direction z in the bulk can be interpreted as the scale Λ on the boundary field theory as well as the radial flow in the bulk is related to the RG flow on the boundary field theory [4]. In this review, I will briefly present the duality of bulk and boundary, the construction of holographic Wilsonian RG on a classical gravity regime.

2 Methods

In this section, I will introduce the methods to set up holographic Wilsonian RG. To begin with, I will present the bulk geometry and the boundary field theory used in the setup[4].

2.1 The Geometric Picture of Bulk & Boundary

2.1.1 Relation Between the Bulk & Boundary

Maldacena's work on AdS/CFT presents that the radial coordinate in the AdS bulk space is associated with the energy scale on the boundary field theory[6]. Susskind and Witten presented[7] that the high-energy theory, ultraviolet(UV) region[1] in the boundary theory is associated to the low-energy, infrared(IR) region in the AdS bulk, and vice versa. Hence we can explain UV-IR connection in terms of the bulk-boundary correspondence. The figure 1 represents the boundary theory(top) with the energy scale Λ, and the bulk geometry(bottom) with the radial coordinate z. As the energy scale in the boundary decreases from $\Lambda_0 > \Lambda' > \Lambda$, the radial coordinate in the bulk increases from $(z = \epsilon_0) < z < (z = \epsilon)$. The boundary is the limit where $z \to 0$. Here, Λ_0 and ϵ_0 are the cut-off for the energy scale in the boundary theory and the radial coordinate in the bulk. I will revisit later how the cut-off scale is determined.

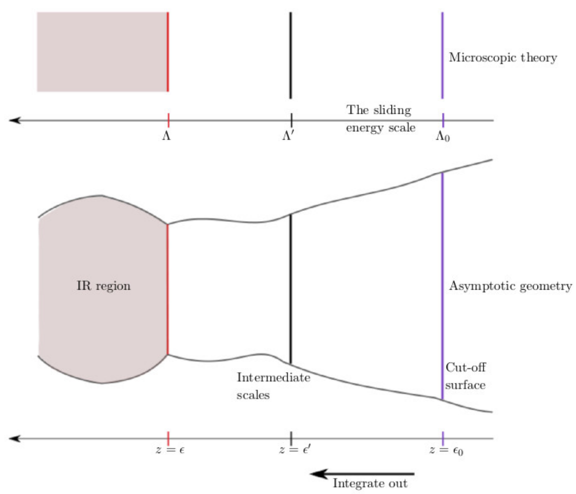

Figure 1: The top is a figure to describe the boundary field theory with the energy scale Λ, and the bottom corresponds to the bulk with the radial scale z. The red-shaded area is where the energy scale is small(IR region). As we integrate out the high energy part, the energy scale goes from Λ_0 to Λ', and eventually Λ at the IR region.

[1]The terminology of ultraviolet(UV) and infrared(IR) is originated from the black body radiation.

2.1.2 $\mathcal{N}$-4 Super Yang Mills Theory

To study the gravity in the bulk, we use $\mathcal{N}$-4 super Yang Mills theory[2]. The $\mathcal{N}$-4 super Yang Mills theory is a 4 dimensional boundary theory, and the theory is dual to type IIB string theory on the $AdS_5 \times S^5$ bulk geometry. By truncating to the lower energy regime, which have only massless fields in the type II string theory, which is supergravity. The supergravity provides an approximation for the gravity in the low energy regime. By taking the large N limit($N \to \infty$), the non planar Feynman diagrams are suppressed by powers of $1/N$. Thus we get the classical gravity approximation[8].

2.1.3 Asymptotically AdS bulk space

As explained in 2.1.2, we are interested in the AdS geometry, and the asymptotic AdS geometry to study the effects of RG. I will closely follow Faulkner's paper[4]. To study the geometry, let's introduce the metric

$$ds^2 = g_{MN}dx^M dx^N \equiv -g_{tt}dt^2 + g_{ii}d\vec{x}^2 + g_{zz}dz^2 \tag{1}$$

g_{MN} only depends on z. By separating the z and the Euclidean part(t, x_i), one can get the equation on the right hand side. The metric components g_{tt}, g_{ii}, and g_{zz} are determined by the choice of the coordinate. In any case, g_{tt} is a monotonically decreasing function of z. The asymptotically AdS can be calculated from the limit $z \to 0$. In contrast, as we go to the interior region of the bulk, we increase z. The time interval Δt at the boundary is shifted as the local proper time$\Delta\tau \simeq \sqrt{g_{tt}}\Delta t$. Therefore, we expect the proper time decreases in the deeper interior bulk. Motivated by this, we can expect lower energy(IR) of the boundary theory as we go deeper into the bulk geometry, and high energy(UV) of boundary is related to the outer region of the bulk. From this relation, we find the UV-IR connection in terms of the bulk-boundary correspondence, and choose z coordinate as the energy scale of the boundary theory.

* Note: We can introduce the ADM formalism to study the radial flow of the bulk as we set the z direction to the energy scale of the boundary. The work on the radial flow by ADM formalism is done by Boer and Verlinde[10]

2.2 Holographic Wilsonian RG Formalism

2.2.1 The Effective Action

To begin with, we first set a cutoff, and integrate out the UV region in the boundary to get an effective field theory for the boundary theory.

Choose a boundary at $z = \epsilon$ in the bulk. Then we can define the bulk action with a scalar field ϕ as following

$$S = \Big(\int_{z>\epsilon} d^{d+1}x\sqrt{-g}\mathcal{L}(\phi, \partial_M\phi)\Big) + S_B[\phi, z = \epsilon] \tag{2}$$

[2]$\mathcal{N}$ is the number of spinor supercharges.

Here, the bulk action is separated into two parts. The integration of Lagrangian $\mathcal{L}$ inside the cutoff region in bulk($z > \epsilon$), and the boundary action in the bulk S_B. The above action is given by integrating out ϕ degrees of freedom in the region $z < \epsilon$. Later, one will see that the effective action $S_B$in the bulk is dual to the Wilsonian effective action I_{UV} of the boundary field theory.

2.2.2 Flow Equation

So far no condition has been imposed on the boundary scale ϵ. Therefore, for the arbitrary scale ϵ, one get the flow equation

$$0 = -\int_{z=\epsilon} d^d x\sqrt{-g}\mathcal{L} + \partial_\epsilon S_B[\phi, \epsilon] + \int_{z=\epsilon} d^d x \frac{\delta S_B}{\delta\phi(x)}\partial_z \phi(x) \tag{3}$$

where the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{4}$$

The equation of motion can be calculated by implementing (4) into (2) and varying the action.

$$\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g})g^{MN}\partial_N\phi) - \frac{\partial V}{\partial\phi} = 0 \tag{5}$$

with the boundary conditions ($z = \epsilon$)

$$\Pi = \frac{\delta S_B}{\delta\phi}, \quad \Pi \equiv -\sqrt{-g}g^{zz}\partial_z\phi \tag{6}$$

The Π is the canonical momentum along the z direction. Coming back to the flow equation, using 6,

$$\partial_\epsilon S_B[\phi, \epsilon] = -\int_{z=\epsilon} d^d x Pi\partial_z\phi - \sqrt{-g}\mathcal{L}) = -\int d^d x\mathcal{H} \tag{7}$$

One get the Hamilton-Jacobi equation, and the equation shows that the flow is generated by the Hamiltonian. Using 4 and 6, one get more explicit expression for the flow equation

$$\sqrt{g^{zz}}\partial_\epsilon S_B[\phi, \epsilon] = -\int_{z=\epsilon} d^d x\sqrt{-\gamma}\big(\frac{1}{2\gamma}\big(\frac{\delta S_B}{\delta\phi}\big)^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\big) \tag{8}$$

where $\gamma \equiv \det g_{\mu\nu} = gg^{zz}$

2.2.3 Choice of Cutoff ϵ

To describe the physics at the fixed point, one use the generating functional with the bulk action and the counter-term

$$e^{I[J]} \equiv \left\langle e^{\int \mathcal{J}\mathcal{O}} \right\rangle = \lim_{\epsilon\to 0} e^{S_0[\phi_c, z\geq\epsilon] + S_{ct}[\phi_c, z=\epsilon]} \tag{9}$$

S_0 is the bulk action, and S_{ct} is the counter-term of the action. ϕ_c is the classical field. Compared to the fixed point, one can express the generating functional with boundary action

$$e^{I[J]} = e^{S_0[\phi_c, z \geq \epsilon] + S_B[\phi_c, z=\epsilon]} \tag{10}$$

If we start the initial value of ϵ at 0, we can get the above generating functional for any ϵ. To expand the correlation function in terms of the low frequency ω or small momentum k, the generating functional with the boundary action (10) is simpler to calculate than (9). *We choose to put the cutoff $z = \epsilon$ where we can not analytically expand the bulk action S_B in terms of ω or k.* The lose of analyticity means the loss of the degrees of freedom as we integrate out the geometry.

2.3 Flow Equation and CFT Deformation

The boundary action in the bulk $S_B[\phi, \epsilon]$ is dual to the effective action on the boundary $I_{UV}[\Phi, \Lambda]$. By applying the alternative quantization of ϕ, one can simplify the relation between these two effective actions. I will first present how one can associate the two actions. Later I will review the CFT deformation in terms of double-trace coupling.

2.3.1 Field Quantization and Power Expansion

The bulk scalar field ϕ can be quantized into two ways - one is the standard quantization, and the other is the alternative quantization. In the first one, Dirichlet boundary condition is imposed on the field ϕ in the AdS space. Neumann boundary condition is imposed on the bulk field ϕ in the latter case. Applying the alternative quantization on the bulk field, the scalar field ϕ is associated with the expectation value of the dual single-trace operator $\mathcal{O}$ in the boundary field theory. If one expand the boundary action S_B in terms of the field ϕ, one will have terms of order ϕ, ϕ^2, ..., ϕ^n. From the above correspondence, each term corresponds to $\mathcal{O}$, $\mathcal{O}^2$, ..., $\mathcal{O}^n$. Up to some renormalization, we find that S_B and I_{UV} are related by a Legendre transformation.[**?**] The explicit calculation is given in the appendix of [4].

2.3.2 Free Massive Scalar Case

To provide an example, we introduce the free massive scalar fields to the bulk theory

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{11}$$

Plug the Lagrangian into the boundary action S_B, and expand it in the momentum space,

$$S_B[\epsilon, \phi] = \Lambda(\epsilon) + \int \frac{d^dk}{(2\pi)^d}\sqrt{-\gamma}J(k,\epsilon)\phi(-k) - \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\sqrt{-\gamma}f(k,\epsilon)\phi(k)\phi(-k) \tag{12}$$

Here, $J(k)$ is related to the coupling for the single-trace operator related to $\mathcal{O}$. Similarly, $f(k)$ is related to the coupling for the double-trace operator $\mathcal{O}^2$. During the expansion, the following notations were used.

$$k_\mu = (-\omega, k_i), \quad d^d k = d\omega d^{d-1} k_i, \quad k^2 \equiv \Sigma_i k_i^2, \quad k^\mu k_\mu = -g^{tt}\omega^2 + g^{ii} k^2 \tag{13}$$

Plugging the boundary action given in the (12) into the flow equation (8), one get the equations for the cutoff energy Λon the boundary, f, and k at the boundary $z = \epsilon$.

$$\frac{1}{\sqrt{-\gamma}}\sqrt{g^{zz}}\partial_\epsilon \Lambda = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} J(k,\epsilon) J(-k,\epsilon) \tag{14}$$

$$\frac{1}{\sqrt{-\gamma}}\sqrt{g^{zz}}\partial_\epsilon (\sqrt{-\gamma} J(k,\epsilon)) = -J(k,\epsilon) f(k,\epsilon) \tag{15}$$

$$\frac{1}{\sqrt{-\gamma}}\sqrt{g^{zz}}\partial_\epsilon (\sqrt{-\gamma} f(k,\epsilon)) = -f^2(k,\epsilon) + k^\mu k_\mu + m^2 \tag{16}$$

With the given equations, we can finally calculate the renormalized couplings.

2.3.3 Flow Equation

As I stated at the beginning, we are studying the classical gravity approximation. Therefore, the field ϕ and the corresponding momentum Π should satisfy classical equation of motion. the equation of motion is

$$\partial_z \phi = -\frac{g_{zz}}{\sqrt{-g}}\Pi \tag{17}$$

$$\partial_z \Pi = -\sqrt{-g}(k_\mu k^\mu + m^2)\phi \tag{18}$$

The couplings that satisfy the above equations are given by

$$f = -\frac{\Pi_s}{\sqrt{-\gamma}\phi_s}, \quad J = \frac{1}{\sqrt{-\gamma}\phi_s} \tag{19}$$

With a given initial couplings f_0, J_0 at the boundary $z = \epsilon_0$, we find

$$\sqrt{-\gamma} J(\epsilon) = \frac{\sqrt{-\gamma_0} J_0}{u(\sqrt{-\gamma_0} f_0) + v} \tag{20}$$

$$\sqrt{-\gamma} f(\epsilon) = \frac{\gamma(\sqrt{-\gamma_0} f_0) + s}{u(\sqrt{-\gamma_0} f_0) + v} \tag{21}$$

where γ_0 is the value of γ defined at the boundary $z = \epsilon_0$. and r, s, u, v are defined by the matrix

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} = M(\epsilon) M^{-1}(\epsilon_0) \tag{22}$$

where M is defined by

$$M(z) \equiv \begin{pmatrix} -\pi_1(z) & -\pi_2(z) \\ \phi_1(z) & \phi_2(z) \end{pmatrix} \tag{23}$$

the $\phi_1(z), \phi_2(z)$ are the two independent solutions, and they form a basis. $\pi_1(z), \pi_2(z)$ are the canonical conjugate momenta for each of the field.

2.3.4 Example: Flow of Double-Trace Couplings in the Vacuum

The deformation of the boundary theory corresponds to the continuum limit of the coupling transformation. Here, we are going to use the vacuum example to derive the double-trace couplings. In the vacuum, i.e., zero momentum pure AdS space, the metric is defined as

$$ds^2 = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{24}$$

Plugging the metric into (14), we get

$$\epsilon\partial_\epsilon f = -f^2 - \Delta\Delta_- + df \tag{25}$$

with the notation

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2}, \quad \Delta_= d - \Delta \tag{26}$$

For simplicity, we will set $R = 1$. As mentioned in the previous section 2.3.1, the two different quantizations leads to different duality between the bulk field and the boundary field operator. In the standard quantization(Dirichlet boundary condition), ϕ is dual to $\mathcal{O}_+$, and the dimension of the operator is Δ. For alternative quantization(Neumann boundary condition), ϕ is dual to $\mathcal{O}_-$, and the dimension of the operator is Δ_-. Using the notation assigned above, we reach to the conclusion that the double-trace coupling f in the alternative quantization has dimensions $2\nu(-2\nu)$. By writing $f = \bar{f} + \Delta_-$, find

$$\epsilon\partial_\epsilon \bar{f} = -\bar{f}^2 + 2\nu\bar{f} \tag{27}$$

And the equation is identical to the double-trace β-function found in the field theory [**?**].

3 Discussion

In this paper, I have introduced the holographic Wilsonian RG. First, we use the supergravity in the classical sense. This classical gravitational theory in the AdS/CFT space bulk is dual to the $\mathcal{N}$-4 super Yang Mills theory on the boundary of the space. Promoting the UV-IR connection, we can integrate out the geometry of the bulk space, leaving $z > \epsilon$. This region has a duality with the boundary theory with energy lower than Λ. The alternative quantization on the bulk field theory ϕ provides the relation that the bulk field theory ϕ is the Legendre transform of the boundary field operator $\mathcal{O}$. Using this relation, we can see the flow equation of the bulk theory to the RG flow of the couplings. The double-trace coupling is especially significant as it is associated with the deformation of the action.

This paper succeeded in constructing a holographic Wilsonian RG to the classical low energy gravity approximation. Compared to old literature[10] that the renormalized theory is non-local and depend on the entire bulk space, the Wilsonian approach determines that the new region after the high momenta geometry is integrated out is independent of integrated out region.

In the review, we used the free massive scalar field. However, in other cases such as vector fields, there can be a region of gapless degrees of freedom in the interior bulk. If this is the case, we can not simply make the statement that integrating out the bulk degrees of freedom is identical to integrating out boundary degrees of freedom at the cutoff scale. In this case, we can still expand the effective action of the boundary theory I_{UV} with a power series of local operators, but we can not expand the bulk boundary action S_B. The boundary action S_B depends on all regions in the bulk. If the gapless states exist in the bulk, one should separate the gapless modes out to calculate the action in a power series expansion.

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