Renormalization flow of the Anderson localization transition

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1 Introduction

What are the transport properties of a non-interacting wave in a disordered medium? This question is so simple that it is hard to imagine that no one thought to ask it before the second half of the twentieth century, but modern study begins with Anderson, in the context of electrons in a disordered solid [1]. As he first showed, there is the possibility of a disorder-driven phase transition. At a critical amount of disorder, eigenfunctions at a given energy change from being extended across the entire system, resulting in conduction for the case of the Fermi energy in an electronic solid, to eigenfunctions that are exponentially localized in space, leading to an electronic insulator [Fig. 1]. Decades of intensive study, including important contributions by at least three Nobel laureates in condensed matter physics, have since refined our understanding of this transition.

Eigenstates that are localized by disorder take the form:

$$\psi_n(r) \sim e^{-|r|/\xi_n} \tag{1}$$

where ξ_n is the *localization length* for this eigenstate. An unavoidable complication in a disordered system is that the precise value of ξ_n will depend on the instance of disorder used. Accordingly, we will always look at ξ, and other properties, as averaged over many instances of disorder generated with the same statistical properties. In this case, ξ should only be a function of the eigenenergy and may be conjectured to have some power law scaling near the transition:

$$\xi(E) \sim (E_c - E)^{-\nu}, \tag{2}$$

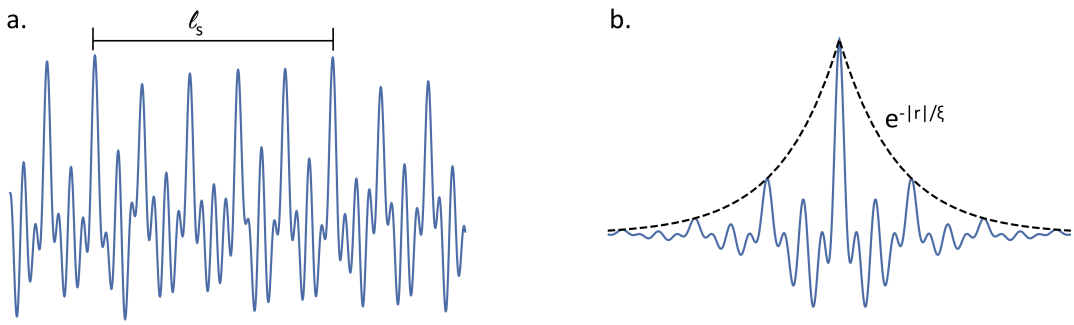

Figure 1: a) In a lattice with weak disorder, the eigenstates may be modified from periodic Bloch waves on the scattering length scale l_s without losing their extended character. b) In sufficiently strong disorder, the eigenstates will become exponentially localized with localization length ξ.

as may the conductivity of an electronic solid:

$$\sigma(E) \sim (E_c - E)^t \tag{3}$$

In fact, these scaling laws are related. Conductivity, when scaled by the fundamental conductance $G_0 = e^2/\hbar$, has units of length divided by cross-sectional length, or $[\sigma/G_0] = [\ell]/[\ell]^{d-1} = [\ell]^{2-d}$. So, supposing that near the transition the only important length scale can be ξ, this implies that $\sigma \sim \xi^{2-d}$ and therefore $t = (d-2)\nu$. Therefore, the central questions in the study of Anderson localization are the value of E_c, known as the *mobility edge*, and the value of the critical exponent ν.

Anderson localization is not a thermodynamic phase transition. It does not cause a change in the local density of states of the system, and therefore has no experimental signature in the free energy (more properly, the ground state energy) or its derivatives [2]. Still, many of the ideas and techniques applied to this transition are taken from the study of thermodynamic phase transitions. ξ is similar to the correlation length. There is no obvious order parameter, although sometimes ξ is used for this as well. Since this is a non-interacting problem one may consider it as a phase transition at the level of a single eigenstate. When applied to electronic systems it is $\xi(E_F)$ that will determine whether the material ground state is insulating (for finite ξ) or conducting (for infinite ξ).

Like many thermodynamic transitions, it was quickly apparent that Anderson localization depends strongly on dimensionality. In one dimension, one may show under

pretty general circumstances that all states are localized with infinitesimal disorder, with a localization length comparable to the microscopic scale of the disorder [3]. In three dimensions (and above) this is presumably not true. The two-dimensional case was therefore expected to be marginal, as suggested by the $(2-d)$ dependence of the conductivity.

The microscopic Hamiltonian first used by Anderson, and also in many subsequent studies, has the following form:

$$\mathcal{H} = -t \left(\sum_{\langle i,j \rangle} |i\rangle\langle j| + h.c. \right) + \sum_i \epsilon_i |i\rangle\langle i| \tag{4}$$

This is a usual tight-binding model with nearest-neighbor hopping controlled by t, and an additional random energy cost to each site, whose values should be drawn from some specific distribution. I have written it in first-quantized notation to emphasize that we are interested in single-particle physics. However, as we will see, many theoretical approaches do not start from the microscopic Hamiltonian but instead at some intermediate scale. Also, I will note in passing that this Hamiltonian, and every other result I will discuss, obeys time-reversal symmetry, which is an important prerequisite for the coherent backscattering that underlies Anderson localization [4]. The fate of Anderson localization in a system of magnetic impurities or another time-reversal-breaking mechanism is a fascinating question that I will not attempt to address.

2 Scaling theory of AL

The most influential application of RG concepts to Anderson localization is the scaling theory of localization as introduced by the "gang of four" [5]. It isn't really a full RG analysis à la Wilson, but closer to the Kadanoff block spin argument. The authors make somewhat vague arguments about the one-parameter scaling of the conductance of a sample as blocks are added together, and then go to the large-sample limit.

The property of study is the dimensionless conductance, $g = G/G_0$. If a block of material with side L has a conductance $g(L)$, how does the conductance change if n of these pieces are added together to make a larger block? We imagine that the initial L is at some mesoscopic length scale that is at least as large as the scattering length l_s of the wave in the disordered medium, so that the dynamics are diffusive.

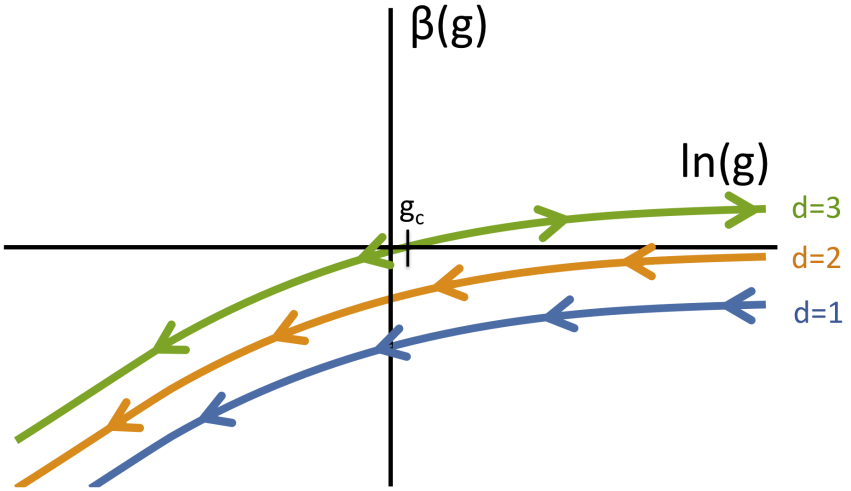

Figure 2: $\beta(g)$ predicted by scaling theory in one, two, and three dimensions. Arrows indicate the direction of flow, with left being the insulating $g = 0$ fixed point and right the conducting $g \to \infty$ fixed point. In three dimensions there is a critical fixed point at g_c.

Two limiting behaviors are important: for disorder insufficient to cause localization, we should recover Ohm's law: $\lim_{L\to\infty} g(L) = \sigma L^{d-2}$, where the intensive conductivity σ is taken to be a constant parameter. For disorder that does cause localization, we should get a form like $\lim_{L\to\infty} g(L) \sim e^{-L/\xi}$.

The crucial step then made by Abrahams et. al. was to claim that the conductance at one scale should only depend on the conductance at the previous scale and the scale itself. That is, $g(nL) = F(n, g(L))$. Taking the continuous limit of this scaling, this means that

$$\begin{aligned} g(L+\delta L) &= F(1+\frac{\delta L}{L}, g(L)) \\ &\approx g(L) + \frac{\delta L}{L}\left.\frac{\partial F(n,g)}{\partial n}\right|_{n=1} \\ \frac{dg(L)}{dL} &= \frac{g(L)}{L}\left(\frac{1}{g(L)}\left.\frac{\partial F(n,g(L))}{\partial n}\right|_{n=1}\right) = \frac{g(L)}{L}F_2(g(L)) \end{aligned} \tag{5}$$

The key result is then that the beta function, $\beta(g) = d\ln g/d\ln L = (dg/dL)(L/g)$, is only a function of g. So, from the limiting behaviors above one then assumes that

for small g the dependence is $\beta(g) \sim \ln(g)$, for large g it is $\beta(g) \sim (d-2)$, and that β monotonically interpolates between the two. A schematic plot of the resulting $\beta(g)$ is shown in Fig. 2. For a negative β, the conductance of the system decreases as the system size increases, and the system is said to be insulating or localized. For a positive β, conductance increases and it is conductive or extended. Therefore, this scaling predicts that in one and two dimensions the system is always localized, although only marginally so in the 2D case, while in 3D there is a localized-delocalized transition at some critical initial conductance. The slope of the line at g_c can be shown to be an estimation of $1/\nu$ [3].

In close analog to the study of Landau critical phenomena, this analysis may be extended perturbatively in two ways: by taking a higher-order calculation of the scattering, and by expanding about a critical dimension. To do the first, one must go back to the quantum transport calculations that lead to Ohm's law and find the next-order effect for weak scattering. This turns out to modify the beta function at high g to

$$\beta(g) = (d-2) - a/g, \tag{6}$$

with some constant a. In two dimensions, this means that at large g (weak scattering) conductance decreases as $g(L) = g(l_s) - G_0 \ln(L/l_s)$, where l_s and $g(l_s)$ are the initial values of the system size and conductance near the scale of the scattering length. Taking the localization length as the scale at which the conductivity vanishes, this means that the localization length increases exponentially with the scattering mean free path, with the estimation working out to be $\xi(k) = l_s e^{\pi k l_s/2}$ for a particle with wavenumber k [3]. This exponential dependence has made experimental verification of the scaling theory in two dimensions challenging, as we will see.

To investigate the phase transition in $d = 3$, the usual starting point is to expand in $d = 2 + \epsilon$. Since $[\sigma] \sim [\ell]^{2-d}$, this is analogous to the $4 + \epsilon$ expansion applied to the Landau quartic coupling constant $[u_0] \sim [\ell]^{d-4}$. This gives a critical conductance of $g_c = a/\epsilon$, for small distance away from the critical point $\beta(g_c + \delta g) = \delta g(\epsilon/a)$, and therefore the prediction that $\nu = 1/\epsilon$. Taking $\epsilon = 1$, one then predicts that in three dimensions $\nu = 1$.

3 Real-space renormalization

As I have mentioned, the original scaling argument, while important, was fairly non-rigorous in a similar way as Kadanoff's block spin argument. Furthermore, it did not use very much of the machinery of RG (perhaps because it was still being

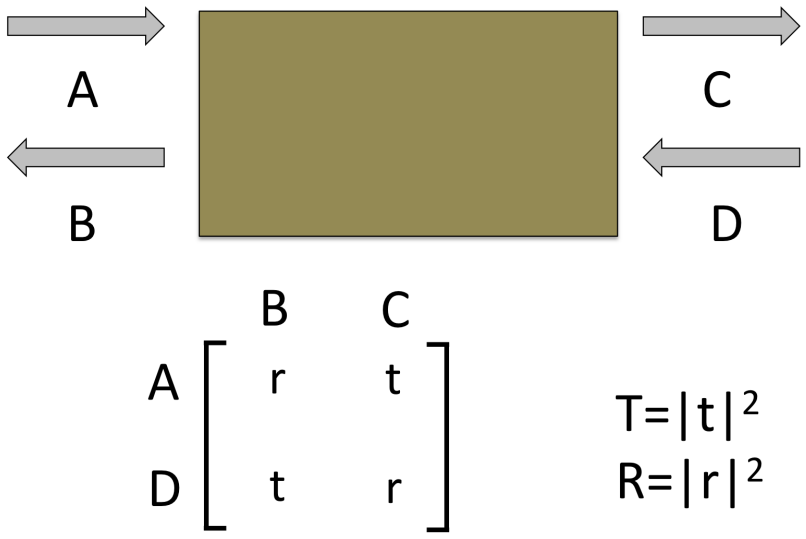

Figure 3: The fundamental object for the network-analysis RG of Anderson localization. Each block has two input and output channels, which are constrained by unitarity and time-reversal symmetry to be connected by the two complex amplitudes r and t.

developed). It is therefore useful to mention a slight refinement developed afterwards by Anderson and Shapiro [6, 7].

In this analysis, one starts in a similar position as the original scaling argument, at some intermediate length scale at which the dynamics are already diffusive. Then one preforms a network analysis of a series of blocks with some reflectance and transmittance [Fig. 3]. These blocks are strung together to make a 1D chain with some overall transmittance, then many of these chains are added in parallel to make a d-dimensional system. Disorder is incorporated by assuming that the phase between each block varies randomly, and averaging over the phase.

For a single block with transmission and reflection T_1, R_1, define the conductance as

$$g^{-1}(1) = \frac{R_1}{T_1} \tag{7}$$

This can be shown to line up with the usual definition of conductance. For example, Ohm's law is recovered in the limit of no interference. Working out the

composition law for transmission from two of these blocks in series, one finds that

$$t(2) = t_1 t_2 \sum_{n=0}^{\infty} (r_1 r_2)^n = \frac{t_1 t_2}{1 - r_1 r_2}, \tag{8}$$

where each term in the infinite sum represents one set of back-reflections between the two blocks. Disorder averaging is done by assuming that $\langle r_1 r_2 \rangle = 0$ due to the random average phase between the two blocks, which leads to

$$T(2) = \left\langle |t(2)|^2 \right\rangle \approx T_1 T_2 \tag{9}$$

For n of these blocks in series, and using $1 + g^{-1} = (T + R)/T = 1/T$, one finds that

$$g^{-1}(n) = (1 + g^{-1}(1))^n - 1 \tag{10}$$

Combining n of these chains in parallel along d dimensions gives the final discrete form for the RG relation:

$$g^{-1}(n) = n^{1-d}[(1 + g^{-1}(1))^n - 1] \tag{11}$$

Or for an infinitesimal scale change of δn:

$$g^{-1}(1 + \delta n) = g^{-1}(1) + \delta n((1 + g^{-1}(1)) \ln(1 + g^{-1}(1)) - g^{-1}(1)(d - 1)) \tag{12}$$

This recursion has trivial fixed points at $g^{-1} = 0$ and ∞, and for $d > 2$ it has an additional non-trivial fixed point. Further analysis gives most of the results already shown above, but one modification is that in addition to the prediction for $\nu = \epsilon^{-1}$ at $d = 2 + \epsilon$, one can also calculate it numerically at $d = 3$ for a prediction of $\nu = 1.68$. This analysis may also be extended to look at the interplay between quantum localization from microscopic disorder and classical localization from a percolation transition due to large-scale disorder, and construct a flow diagram incorporating both effects [7].

4 Non-linear sigma model

The arguments given above still form the core of theoretical treatments of Anderson localization, but it is worth mentioning briefly a major technical advance carried out over the last thirty years. Starting with Wegner [8], a mapping was found from the Anderson-localization Hamiltonian to a known field theory, the non-linear sigma model (NLSM). This has allowed the application of various technical tools developed for supersymmetry to the problem [4]. One significant advancement due to these techniques is a greater understanding of the role of fluctuations, which up until now we have ignored in favor of disorder averages. However, analyses of these models have shown that 'typical' wavefunctions can be very different than the average, with system properties being heavily influenced by rare configurations [9]. However, analyses using this model still suffer from some of the limitations of the simpler RG described above- for example, it is also limited to $d = 2 + \epsilon$, which has limited its predictive ability.

5 Experiment/ Simulation results

Anderson localization has been the subject of more numerical and experimental investigations than I could hope to summarize. So I will only touch on a few relevant results in each:

It is no accident that the study of Anderson localization has come of age along with computational physics. Much of our most reliable knowledge about Anderson localization is from simulation. For example, the validity of one-parameter scaling has been investigated and found to be very accurate for the Hamiltonian (4) [Fig. 4] [10]. In addition, the exponent ν has been calculated with increasing accuracy, and is now known to be $\nu = 1.58 \pm 0.01$ [11].

Meanwhile, many experiments looking for Anderson localization have been preformed, in experimental systems ranging from condensed matter [12] to ultracold atoms [13, 14] to classical ultrasound and light scattering [15, 16, 17]. All have their challenges. In electronic materials one might not know or be able to control the microscopic disorder, and electron-electron or electron-phonon interactions may influence the results. In ultracold atoms limits on the sample size and observation time can make it difficult to determine if the system is localized near the transition, and as in condensed matter observations are generally an average over a thermodynamic ensemble of particle energies. In light and ultrasound experiments there are challenges getting to the sufficiently strong scattering regime while minimizing absorption [18]. As a result, the experimental cutting edge seems to be rather primitive:

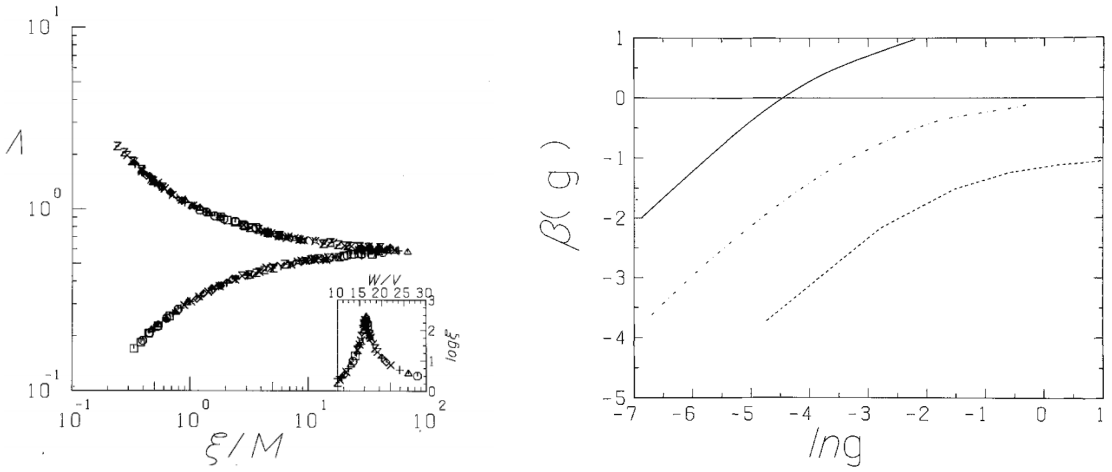

Figure 4: Results of numerical investigation of the scaling theory, from [10]. Left: Scaling of the localization length in three dimensions, showing data collapse above and below the transition. Right: Resulting numerical calculation of $\beta(g)$ for $d = 1, 2,$ and 3.

representative results are a fairly confident observation of a localization transition or lower-dimensional crossover, and measurement of the trajectory of the mobility edge (e.g. [19]). Attempts to measure ν, which as far as I know have all been in condensed matter systems, have reported values dependent on the material generally near $\nu = 1.0$ or $\nu = 0.5$, with the former believed to be due to Anderson localization and the latter due to other material effects [20].

The two-dimensional case deserves special mention. Historically, one of the central questions related to Anderson localization is whether a metal-insulator transition exists in two dimensions. Prior to the development of scaling theory an argument due to Mott suggested that it should, and additionally that there should be a minimum metallic conductivity below which there is a sharp jump to an insulating state [5]. Scaling theory predicted instead, as we have seen, that a two-dimensional system is fully localized by infinitesimal disorder, and perturbative treatments suggested that interactions should not change this conclusion [21]. However, some early experiments in thin films appear to show a metal-insulator transition, and in addition to not obey single-parameter scaling [22]. More recently, Kravchenko et al have found quite strong evidence of a metal-insulator transition in thin films that exhibits universal scaling. However, they attribute this transition to the strong interactions in their system and believe that in a sufficiently weakly interacting regime the scaling behavior is recovered [23]. Although the scaling theory prediction seems to be widely

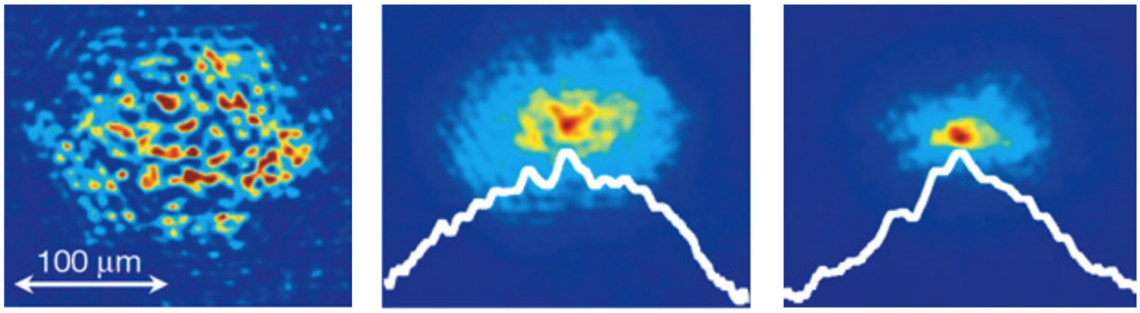

Figure 5: Transverse localization of light propagating through a photonic lattice with increasing disorder, from [24].

accepted, experimental confirmation is likely to wait until a sufficiently sophisticated experiment in a non-interacting system is possible, perhaps an improved measurement of transverse localization in a photonic crystal [24] or a measurement using cold atoms confined to two dimensions [25].

6 Conclusion

Anderson localization, as a minimal example of a phase transition beyond textbook thermodynamics, makes a neat case study to demonstrate the generality of RG ideas. The history of localization research also gives a nice window into how views of RG have developed over time, with the early papers showing how some of the modern understanding wasn't quite there yet and later ones using it in very sophisticated ways. Nonetheless, the challenges of the localization problem remain. Even after more than fifty years of intensive study on all fronts, results from RG, numerics, and experiment have yet to really converge.

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