Blackhole Thermodynamics and Hawking-Page Phase Transition

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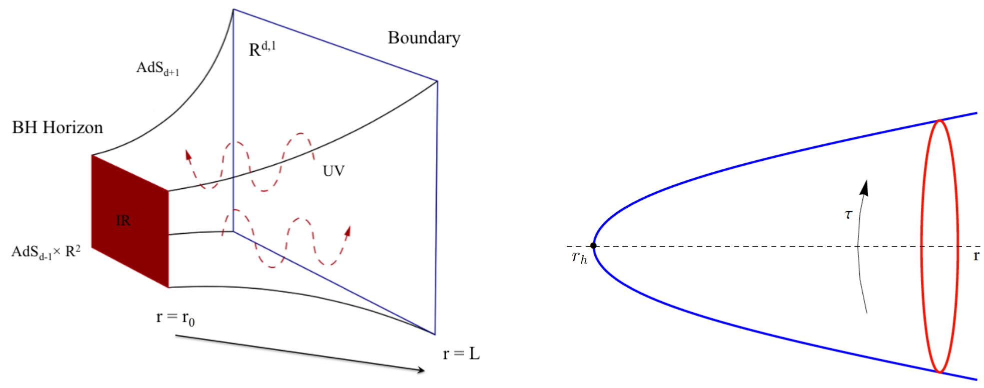
Abstract

We briefly review general relativity, present some basic ideas of AdS/CFT correspondence, thereby motivating the importance of anti de Sitter (AdS) spacetime. We show how to compute thermodynamic quantities such as temperate, free energy for a given background geometry. By computing free energy, we show in AdS space, if a black hole is smaller than a critical size then it becomes unstable and evaporates into thermal radiation. This first order phase transition is driven by breaking of conformal symmetry of AdS.

1 Introduction

With increasing temperature, the thermal fluctuations start increasing in a system, resulting in rearrangement of zero-temperature degrees of freedom in such a way that it might sometimes result in a completely different phase. Ice melting into liquid water, or ordered to disordered phase transition in an ising lattice, a normal metal becoming a superconductor or an insulator etc are some familiar examples of classical phase transitions. These degrees of freedom do not necessarily have to be of matter fields. General relativity allows us to treat space-time geometry dynamically, meaning geometry can be seen as an emergent phenomenon of matter (stress tensor). Here we study a phase transition in the spacetime geometry, which is a transition from a black hole (BH) solution to a no-black hole solution (called *thermal* spacetime). This is known as Hawking-Page (HP) phase transition [1].

The reader must be warned that this transition is significantly abstract to study or verify in any of our laboratory setups. The phase transition occurs in a special kind of spacetime called anti de Sitter (AdS) space (see Fig. 1a), which



a. Anti de Sitter Space **b.** Circle of Imaginary Time.

Figure 1: (a) AdS spacetime with a BH and boundary; (b) The Euclidean geometry with compact time dimension, with period equal to inverse temperature.

is not a physical spacetime like Minkowski spacetime. Then why should we care? The answer lies in our attempts to understand physics at Planck energy, $E_p = \sqrt{\hbar c^5/G} \approx 10^{19}$ GeV. In other words we want to understand 'how do quantum particles or fields behave under very strong gravity'. In the past two decades, this question has led to a plethora of theories of quantum gravity, such as string theory or loop quantum gravity etc. During these attempts the idea of a holographic universe [2] was developed and independent to the validity of all other theories, the holographic principle of Maldacena [3] (also known as AdS/CFT correspondence) has proven to be a powerful calculation tool [4]. Here CFT stands for a field theory with conformal symmetry, loosely speaking this is equivalent to a scale invariant theory.

The idea of this correspondence is the following. Like a hologram encodes three dimensional data on a two dimensional sheet, AdS/CFT conjectures that our four dimensional universe (Minkowski space) can be seen as a hologram of a five dimensional universe which has AdS spacetime geometry. To be more precise, if we have a conformal (or quantum) field (Φ) theory described on a d-dimensional physical or Minkowski space, with partition function $\mathcal{Z}[\Phi]$, we can reproduce a lot of the details of this theory by looking at the boundary of a $(d+1)$-dimensional classical field (or string) theory in AdS space,

$$\mathcal{Z}_{CFT}[\Phi] = \exp(-iS^{o.s.}_{bdy}[\Phi, g_{\mu\nu}])\,. \tag{1}$$

Here the field theory is described by the action $S[\Phi, g_{\mu\nu}]$ and $S^{o.s.}_{bdy}$ is the sad-

Boundary (operator)	**Bulk (field)**
Stress tensor $T_{\mu\nu}$	Metric g_{ab}
Global current J_μ	Maxwell field A_a
RG flow	Radial coordinate
Bosonic operator	Klein-Goldon field
Fermionic operator	Dirac field
Scaling dimension of operator	Mass of the field
Global symmetry	Local isometry
Temperature of the field theory	Hawking temperature of BH
Phase transition	Instability of black holes

Table 1: A holographic dictionary: an entry from the left column, which is an operator in a field theory living on the boundary, can be mapped to the corresponding entry (field) on the right column, a quantity that describes the gravity theory on the bulk of AdS.

dle point action (or *on-shell action*), computed at the boundary. A more useful ”dictionary” of correspondence is presented in Table 1. The above relationship makes the study of AdS spaces very promising. Using this correspondence, the HP phase transition was later given field theoretic meaning by Witten, who showed that this transition in AdS_5 space is nothing but a holographic dual description of confinement-deconfinement transition in a $SU(N)$ gauge theories, such as QCD [5]. Later on the relation with Hagadron transition in string theory or Kosterlitz-Thouless transition in condensed matter were also explored [6].

We arrange the term paper in the following manner, in Sec. 2 we briefly review general relativity and discuss some properties of AdS geometry by solving it from vacuum Einstein equation for negative cosmological constant. In Sec. 3 we compute temperature of a background geometry by constructing ’time circle’. In Sec. 4 we compute the free energy for the thermal AdS and BH-AdS and show that there exists a critical temperature (or a BH radius), where the free energy difference changes sign, signaling a phase transition. Instead of trying to be historically accurate by sticking to the AdS (bulk) description only, we will be referring to the boundary field theory implications of many of the physical quantities as we come across them. We work in $c = \hbar = k_B = 1$ unit.

2 Anti-de Sitter Spacetime

The Einstein equation of general relativity can be see as the equation of motion for the metric tensor, obtained by minimizing the following action written over a $(d+1)$-dimensional spacetime manifold $\mathcal{M}$, with boundary spacetime $\partial\mathcal{M}$ [7].

$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d}x\sqrt{\gamma}\left(-2K+\frac{4L\Lambda}{d}\right). \tag{2}$$

The first piece in the action is called the Einstein-Hilbert action with a cosmological constant, Λ. The metric tensor on $\mathcal{M}$ is $g_{\mu\nu}$, which has a determinant g. The second piece is added whenever the spacetime has a boundary (a finite size universe), referred to as the Gibbons-Hawking-York boundary term[1]. Here $\gamma_{\mu\nu}$ is the metric of the boundary $\partial\mathcal{M}$. K is the trace of the extrinsic curvature of the boundary manifold, defined as $K = \gamma^{\mu\nu}\nabla_{\mu}n_{\nu}$, here n_{ν} is the outward normal vector on the boundary. Notice we haven't written any matter action (energy-momentum tensor), hence the equation of motion we obtain will give us vacuum solutions or empty universes. Varying the above action we get the following Euler-Lagrange equation of motion, also known as (vacuum) Einstein equation:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}\,.} \tag{3}$$

Here $G_{\mu\nu}$ is the Einstein tensor. $R_{\mu\nu}$ is the Ricci tensor, which is obtained from various tensorial operations on the metric tensor. $R = g^{\mu\nu}R_{\mu\nu}$ is called the scalar curvature or Ricci scalar. The simplest solutions to this equation are spacetime manifolds with constant curvature (such as a sphere or a plane):

$$R = 2\Lambda\left(\frac{d+1}{d-1}\right) = \begin{cases}\Lambda = 0, & \text{(flat) Minkowski space}\\ \Lambda > 0, & \text{de Sitter space}\\ \Lambda < 0, & \text{Anti-de Sitter space}\end{cases} \tag{4}$$

The cosmological constant is a measure of vacuum energy of the spacetime, caused by virtual excitations in the vacuum. Our observable universe has a very small positive cosmological constant (an expanding universe), hence in a strict sense it's a de Sitter space. For the purpose of discussing Hawking-Page transition we

[1] A more condensed matter perspective of this term is, this is equivalent to the requirement of adding a Wess-Zumion-Witten (WZW) boundary term to a Chern-Simons topological gauge theory in the bulk, in order to describe the boundary modes or edge states of a Hall system.

restrict our discussion to AdS spaces. With $\Lambda = -d(d-1)/2L^2$, the solution to the Einstein equation [Eq. (3) or (4)] is the AdS$_{d+1}$ metric, given by (in a coordinate system that globally describes AdS)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\vec{x})^2\,, \quad f(r) = 1 + \frac{r^2}{L^2}\,. \tag{5}$$

Here $\vec{x}$ is written to collectively denote all the remaining $(d-1)$ spatial coordinates. L is the radius of curvature, a smaller L means the spacetime has a larger curvature [2]. Notice the negative sign in front of dt^2, this is called Lorentz signature. By performing a wick rotation we can go to an imaginary, $\tau = it$ thereby making this negative time go away, $d\tau^2 = -dt^2$. This signature is called Euclidean signature. We'll stick to this signature for reasons that will be clear in the following section.

A more useful coordinate system is Poincaré coordinates, which covers $r \geq 0$ region of the space. In this system the metric becomes

$$\boxed{ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\left(d\tau^2 + d(\vec{x})^2\right)\,.} \tag{6}$$

In this coordinate system the boundary is at $r \to \infty$ and $r = 0$ is known as the Poincaré Killing horizon. The geometry above or in Eq. (6), have no BH, we refer to them as *thermal AdS*. An important observation should be made here, if we scale $(\tau, \vec{x}) \to \lambda(\tau, \vec{x})$ then by rescaling r as r/λ, the metric can be left invariant. Thus scale invariance (conformal symmetry) is an *isometry* of the AdS space. Thus we can see how AdS/CFT gives a geometrical meaning to the conformal symmetry of the field theory living on the boundary $\partial\mathcal{M} = \{(\tau, \vec{x})\}$.

The Einstein equation also admits a solution that has a Schwarschild BH in the AdS spacetime. The Schwarschild-AdS (S-AdS) background has a metric,

$$ds^2 = \frac{r^2}{L^2}\left(f(r)d\tau^2 + (d\vec{x})^2\right) + \frac{L^2}{r^2 f(r)}dr^2\,, \quad f(r) = 1 - \left(\frac{r_h}{r}\right)^d\,. \tag{7}$$

Here, r is the radial direction, $f(r)$ is called the redshift factor or the emblackening factor. Near the boundary $r \to \infty$, $f(r) \to 1$, the metric looks like AdS metric

[2] A more mathematically accurate meaning of L is, it's the radius of the embedding hyperboloid: $-X_1^2 - X_2^2 + \sum_{i=3}^{d+1} X_i^2 = -L^2$, a Lorentzian analogue of a $(d+1)$-dimensional sphere with two time-like coordinates, (X_1, X_2). Note L should be non-zero and has dimension of length.

in Eq. (6). We say the S-AdS geometry is ’asymptotically AdS’. Note the metric describing a Schwarzschild BH in Minkowski space is given by

$$ds^2 = f(r)d\tau^2 + dr^2/f(r) + r^2(d\vec{x})^2\,, \quad f(r) = 1 - 2GM/r. \tag{8}$$

Near the boundary this becomes $ds^2 = d\tau^2 + dr^2 + r^2(d\vec{x})^2$, which is the flat Minkowski space. We call the geometry is ’asymptotically flat’. In the following section we discuss the thermodynamics of these spacetime backgrounds.

3 Black Hole Thermodynamics

Temperature. The idea of temperature in a field theory emerges when we introduce imaginary time by performing a Wick rotation, $\tau = it$. To describe a theory at temperature T we make this imaginary time periodic by the factor β, where $\beta = 1/T$. In our context, the periodicity we obtain will help us find the temperature of the boundary field theory. Lets consider a generic BH solution

$$ds_E^2 = g_{tt}(r)d\tau^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(d\vec{x})^2\,. \tag{9}$$

The horizon is defined where $g^{rr}(r_h) = g_{tt}(r_h) = 0$. So a near-horizon expansion would be $g_{rr}(r) \simeq g'_{tt}(r_h)(r-r_h)$, $g_{rr}(r) \simeq g'_{rr}(r_h)(r-r_h)$ and $g_{xx}(r) \simeq g_{xx}(r_h)$. Now performing a coordinate transformation $R = 2\sqrt{r-r_h}/\sqrt{g^{rr'}}$ this metric becomes

$$ds_E^2 = dR^2 + \frac{R^2}{L_\tau^2}d\tau^2\,, \quad L_\tau = 2/\sqrt{g'_{tt}(r_h)g^{rr'}(r_h)}\,. \tag{10}$$

This metric is a plane metric with compact τ axis, see Fig. 1b. The compactified time axis (*time circle*) has a (proper length) perimeter $2\pi L_\tau\sqrt{g_{tt}(r)}$, which smoothly shrinks to a point as we approach the horizon. If one does a local measurement, the periodicity of this imaginary time is $2\pi L_\tau$, which is the inverse temperature β.

Using this method if we compute the temperature of a flat space Schawrschild BH [in Eq. (8)] then it becomes (also known as Hawking temperature)

$$\beta = 2\pi L_\tau = 8\pi GM\,. \tag{11}$$

Free Energy. The free energy of a theory is given by, $F = -T\ln\mathcal{Z}$, where $\mathcal{Z}$ is the partition function. Under saddle point approximation, the dominant contribution to the partition function comes from the action [Eq. (2)] written on the manifold, the metric of which solves the Einstein equation [Eq. (3)], or the on-shell action. Note since we want to study the thermodynamics of this theory we would use the compactified imaginary time metric (Euclidean signature). Thus the free energy is given by

$$F = -T \ln \mathcal{Z} \approx T S_{\mathrm{E}}^{\mathrm{o.s.}} \,. \tag{12}$$

As an example, lets compute the free energy of the Schwarzschild BH in AdS_{d+1} spacetime. Substituting the metric in Eq. (7), in the Einstein-Hilbert action [Eq. (2)] we obtain the on-shell action,

$$F_{\text{S-AdS}} = \frac{TD}{8\pi G L^2} \lim_{\rho\to\infty} \left(\int_0^\beta d\tau \int_{r_h}^{\rho} dr \int_{V_D} d\vec{x} \right) = -\frac{L^D V_D}{16\pi G} \left(\frac{4\pi T}{D} \right)^d \tag{13}$$

Here $D = d-1$ is the number of spatial dimensions of the boundary sub-manifold with volume V_D. Note that the topology of this space is crucial in determining the free energy. At the boundary, we saw time is compact, hence it's a circle topology S^1. We can choose the remaining spatial directions on the boundary manifold to be compact, say S^D or a non-compact space extending to infinity in all directions, say R^D. In earlier case we denote the line element as $d\Omega_D$ and the later case as $d\vec{x}$. Since in AdS, with large enough temperature the BH horizon (which has a sphere topology) can be made to coincide with the boundary manifold, only the choice of S^D makes sense. The R^D choice means the BH horizon is spread in all directions, and reducing temperature causes it to recede in r direction, but never vanish, unlike the S^D case. This topology is also called a 'black brane'. So strictly speaking the above computed free energy is that of a S-AdS black brane.

In the following section we compute the free energy for AdS with boundary, $\partial\mathcal{M} = S^1 \times S^D$. We'll see that along the radial direction of S^D, the conformal symmetry of AdS is broken, causing a gapped state (gap $\sim M$) to emerge, which is a black hole solution. This doesn't happen in case of $\partial\mathcal{M} = S^1 \times R^D$, since the conformal symmetry is unbroken.

4 Hawking-Page Phase Transition

In order to see if there is any phase transition possible in AdS space, between a BH solution to a no BH solution, we want to compute the difference in free energies

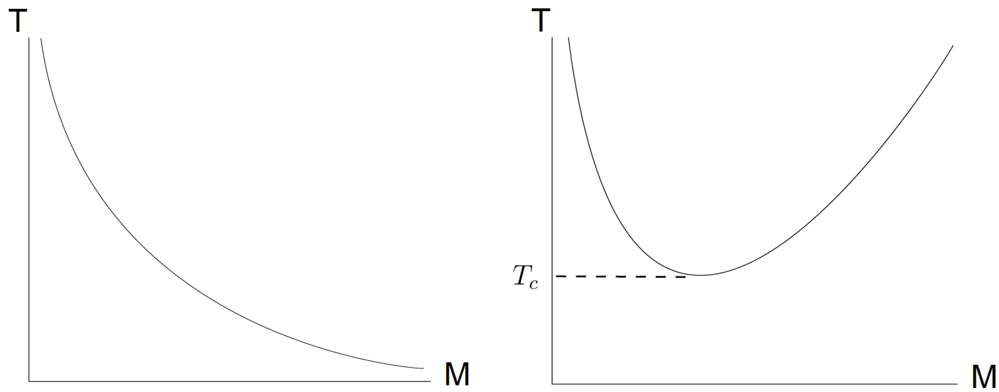


Figure 2: Temperature of a Schwarzschild BH plotted against BH mass M, in flat space (left) and in AdS space (right). Figure adapted from [8].

associated with the respective metrics. In the global coordinate AdS_{d+1} metric is

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_D^2\,,$$
$$\text{where} \quad f(r) = 1 + \frac{r^2}{L^2} - \omega_D \frac{M}{r^{d-2}}\,, \quad \omega_D = \frac{16\pi G}{D V_D}\,. \tag{14}$$

By setting $M = 0$, we can go from (Schwarzschild) BH metric to thermal AdS metric. The horizon of the BH is at r_h, such that $f(r_h) = 0$. Using the methods discussed in the previous section, we can compute the temperature of this BH geometry,

$$\beta = \frac{4\pi L^2 r_h}{d r_h^2 + (d-2)L^2}\,. \tag{15}$$

Notice the difference of this temperature as compared to the temperature of a BH in flat spacetime in Eq. (11). This is plotted in Fig. 2. Unlike the flat space BH, there exists a temperature minima in AdS BH, which signal a phase transition. We'll later see that this is the critical temperature.

The next thing is to compute the temperature in the thermal AdS space, local to a radial coordinate ρ. Since the boundary should not be affected by the BH phase transition we must demand that this local local temperature be equal the BH temperature computed above when ρ reaches the boundary

$$\beta_{loc}(\rho) = \beta \left(\frac{1 + \frac{\rho^2}{L^2} - \omega_D \frac{M}{\rho^{d-2}}}{1 + \frac{\rho^2}{L^2}} \right)^{1/2}\,. \tag{16}$$

Now we compute the difference in free energy (cf. previous section)

$$\beta\Delta F = \frac{D}{8\pi G L^2} \lim_{\rho\to\infty} \left(\int_0^{\beta} d\tau \int_{r_h}^{\rho} r^D dr - \int_0^{\beta_{loc}} d\tau \int_0^{\rho} r^D dr \right) \int_{S^D} d\Omega_D \,,$$

$$\boxed{F_{bh} - F_{th} = \frac{r_h^D V_{S^D}}{d r_h^2 + (d-2)L^2}(L^2 - r_h^2)\,.} \tag{17}$$

Here F_{bh} is the free energy of AdS$_{d+1}$ with the BH solution and F_{th} is the free energy without BH. Note the expectation value of energy is $\langle E \rangle = M$, which is like a gap. This difference when plotted against r_h looks like a mexican hat potential, suggesting energetically favorable BH solutions exist at a finite size. In other words at low temperatures, the horizon radius is small so r_h could be smaller than L, making the thermal AdS a less energetic solution. With rising temperature, a sign change might happen when $r_h > L$, causing a first order (continuous) transition to a BH solution. So a BH with radius small than a critical size can not survive the thermal fluctuation and thermal AdS becomes a more stable solution. (describe the entropy driven argument). The critical size is the AdS curvature radius $r_h = L$. Since the size of the BH can be related to it's temperature via Eq. (15), so the critical temperature associated with this r_h becomes

$$\boxed{T_c = D/2\pi L\,.} \tag{18}$$

Due to space restriction we are refraining from discussing other fixed points in this geometry, which correspond to two differently massive BH solutions. We also avoid discussing entropic issues, which primarily drive such phase transitions. Relations to QCD or hagadron transitions are also omitted.

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