Phase Transitions Methodology Applied to Localization

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Abstract

This paper is intended to explore localization transitions, phenomena where system states become localized due to the presence of disorder. The paper will divulge various results and ideas from the application of traditional phase transitions methods, including the dimensionality dependence from the renormalization group in the metal-insulator transition and the application and subsequent modification to Goldstone's theorem.

1 Introduction

Disorder is a natural topic to investigate if one looks to have grounded theories. All actual existing physical systems contain disorder to some degree, and for this reason alone its effects are worth investigating. And investigation turns up that it plays a key role in a number of phenomena, such as the Kondo problem that determined that how the conductivity of metals varies with temperature is primarily due to free electrons scattering off magnetic impurities. High-temperature superconductors have phase shifts that depends on doping level, where it shifts from a Mott insulator (an insulator with a half-filled band) and disorder can disrupt this evenness of energy levels to allow propagation. It is responsible for metal-insulator transitions, where it can interrupt transporting modes enough to prevent long-distance propagation and destroy conductivity. Disorder is even interesting for cases in which it is an unimportant feature (or perhaps precisely because of this) as in topological phases. The general problem was spotlit by Anderson, who examined states hopping about through transfer coefficients and diffusion processes [4].

Disorder is a difficult topic. By it's very nature it ruins many symmetries, and in the case of quantum disorder where disorder is time-independent, there is no sense of equilibrium so we cannot wait for disorder to "set" and then proceed with an anlysis where perhaps symmetries are restored.

What we study here is a specfic effect of disorder: localization. Disorder breaks translational invariance, and as such it tends to disrupt any free modes (or extended states). One can imagine a smooth surface now made bumpy, disrupting the propagation of modes that would normally go by without dissipating. If extended states correspond to free modes, the other option is localized states, where disorder manages to disrupt propagation enough to confine states. The question is, how much disorder does it take to cause localization?

The investigation of the topic involves many phase transitions techniques. It has several properties of traditional phase transitions at least which encouraged the application of ideas like the renormalization group, critical exponents, an order parameter and goldstone modes (or lack thereof, as we will see). The transition is characterized by a critical energy E_c at which localization sets in, instead of the usual phase transition onset at some critical temperature T_c. A distinction is made by terming this manner of transition a *quantum phase transition*.

2 Order parameter and critical exponents

Like any phase transition we are interested in some sort of measure of how ordered the system is, or how much of the broken phase exists. Here, we move from entirely extended states to introducing localized ones at some critical energy E_c. The logical choice for an order parameter is the density of states, which measures how many states are extended or localized [1] (discussed more in the next section). Despite this measure it provides it is not exactly an order parameter,

as it does not need vanish outside the Goldstone phase. Regardless, we will treat it as one. If the density of states is an order parameter then

$$\rho(E) \sim \left| \frac{E - E_c}{E_c} \right|^{\beta} \tag{2.1}$$

where $(E - E_c)/E_c$ takes the place of a reduced temperature as a measure of approaching the critical point. If $\rho(E)$ is not to vanish as $E \to E_c$, then we require the critical exponent $\beta = 0$. We could not have $\beta < 0$ as well as the order parameter should certainly not diverge, or else it is not in any sense an appropriate order parameter.

We can use this to extract a few other relationships. First

$$\alpha + 2\beta + \gamma = 2 \tag{2.2}$$

means we must have $\alpha + \gamma = 2$. There are also

$$2 - \alpha = d\nu \tag{2.3}$$

$$\gamma = \nu(2 - \eta) \tag{2.4}$$

and combiming these three results gives

$$\nu(d - 2 + \eta) = 0. \tag{2.5}$$

Naturally we expect $\nu \neq 0$ or else the localization length would not diverge and we would not have the transition to extended states. Therefore we get the exact result $\eta = 2 - d$.

3 Alternative to Goldstone's theorem

A usual mark of a phase transition is the appearance of a Goldstone boson. These are massless modes associated with the breaking of a continuous symmetry. The determination of the existence of Goldstone bosons is usually and conveniently made through Ward identities. The following is the argument created by McKane and Stone, who in their process elegantly demonstrated a potential violation of the theorem and pointed out many matching features between localization transitions and conventional phase transitions [1].

To examine localization it makes sense to examine correlators. These objects will inform as to whether states have long-scale behavior or not. We examine the simplest conceivable system where the energy eigenstates follow a Schrödinger equation

$$(-\nabla^2 + V)\psi = E\psi \tag{3.1}$$

where V is to be a simple white noise potential that follows a Gaussian distribution according

to

$$P[V] \propto \exp\left[-\frac{1}{2\gamma}\int d^d x V^2(x)\right]. \tag{3.2}$$

The presumption here is that if this system is to exhibit features of universality, the type of noise and other details are not important.

The retarded correlator must then follow

$$(-\nabla^2 + V - E - i\eta)G(x, y; E + i\eta, V) = \delta(x - y) \tag{3.3}$$

and has the eigenfunction expansion

$$G(x, y; E + i\eta, V) = \sum_n \frac{\psi_n^*(x)\psi_n(y)}{E_n - (E + i\eta)}. \tag{3.4}$$

The density of states is a useful parameter in measuring localization (we will later mention its function as an "order parameter" of sorts). We write the total number of states

$$\rho(E, V)L^d = \sum_n \delta(E - E_n). \tag{3.5}$$

This is directly related to the imaginary part of the Green's function

$$\begin{aligned} \lim_{\eta\to 0^+} \mathrm{Im}\int d^d x G(x, x; E + i\eta, V) &= \lim_{\eta\to 0^+}\int d^d x \sum_n \psi_n^*(x)\psi_n(x)\frac{\eta}{(E_n - E)^2 + \eta^2} & (3.6)\\ &= \pi\sum_n \delta(E - E_n). & (3.7) \end{aligned}$$

Hence

$$\rho(E, V)L^d = \frac{1}{\pi}\lim_{\eta\to 0^+} \mathrm{Im}\int d^d x G(x, x; E + i\eta, V). \tag{3.8}$$

Next, disorder-averaging removes position dependence due to the noise V and so the integral in (3.8) merely gives the volume of the system. Therefore defining disorder-averaged quantities via

$$\overline{F} = \int \mathcal{D}V P[V]F[V] \tag{3.9}$$

we can write

$$\overline{\rho(E)} = \frac{1}{\pi}\lim_{\eta\to 0^+} \mathrm{Im}\overline{G(x, x; E + i\eta)}. \tag{3.10}$$

Individual states for any particular G will extend through the system for some distance $\xi(E)$, termed the localization length. The localization length characterizes states as extended or localized if it is infinite or finite, respectively. This length is analagous to the correlation length of conventional phase transitions. As we approach the critical energy from the localized phase, we must see the localization length $\xi(E)$ diverge to obtain extended states, just as we

would analagously see a correlation length diverge approaching the critical point.

One issue is that although G may extend through the system, the average of G will always be short-ranged. We are essntially considering systems with some random potential V and then averaging over the lot of systems. We see $\overline{G}$ become short-ranged due to unmatching random phases between states with different V copies. To skirt around this complication we can analyze an object without phase dependence, for example $\overline{|G(x,y;E+i\eta)|^2}$. This is an average over the product of the retarded and advanced Green's functions. This quantity will be a crucial part of the Ward identity used for Goldstone's theorem.

If we examine the long-time limit on our correlators, the only states that should be present in our system are the localized ones, while the extended ones have gone off to spatial infinity by now. Hence, our correlator prior to disorder-averaging will depend only upon localized states as

$$\lim_{\eta\to 0^+} \eta|G(x,y;E+i\eta,V)|^2 = \pi\sum_l \delta(E-E_l)|\psi_l(x)|^2|\psi_l(y)|^2 \tag{3.11}$$

where the label l marks localized states. Note that $\eta\to 0$ corresponds to the long-time limit. Next integrate over all space and disorder average to get

$$\pi\overline{\rho_l(E)} = \int d^dy \lim_{\eta\to 0^+} \overline{\eta|G(x,y;E+i\eta)|^2}. \tag{3.12}$$

We will see that this is the key formula that poses an alternative to Goldstone's theorem. We can find a very similar equation for of the density of all states, not just the localized ones. To do so we need only begin with our already proven relationship to the imaginary part of the Green's function. Observing

$$\begin{aligned}
\mathrm{Im}G(x,x;E+i\eta,V) &= \sum_n \psi_n^*(x)\psi_n(x)\frac{\eta}{(E_n-E)^2+\eta^2} &(3.13)\\
&= \eta\int d^dy \sum_{m,n}\frac{\psi_n^*(x)\psi_n(y)\psi_m^*(x)\psi_m(y)}{(E_m-E-i\eta)(E_n-E+i\eta)} &(3.14)\\
&= \eta\int d^dy|G(x,y;E+i\eta,V)|^2 &(3.15)
\end{aligned}$$

we can disorder average before taking the long-time limit to obtain

$$\pi\overline{\rho(E)} = \lim_{\eta\to 0^+}\eta\int d^dy\overline{|G(x,y;E+i\eta)|^2}. \tag{3.16}$$

All the crucial ingredients are established. Now, we see that the disorder averaging and the $\eta\to 0$ limits do not commute. One leaves us with only localized states while the other retains all of them.

Notice we subtly wrote down a Ward identity in (3.15). It is

$$G(x,x;E+i\eta,V)-G(x,x;E-i\eta,V)=2i\eta\int d^dy|G(x,y;E+i\eta,V)|^2 \tag{3.17}$$

which relates the difference in two correlators through a higher-order one. In field theory terms this would be comparing the vertex functions for two distinct fields to a higher order vertex function. If we are in the broken symmetry phase where the density of states is nonzero, we should find one of two possibilities. Either

$$\lim_{\eta\to 0^+}\eta\int d^dy\overline{|G(x,y;E+i\eta)|^2}\neq 0 \tag{3.18}$$

or

$$\int d^dy\lim_{\eta\to 0^+}\overline{\eta|G(x,y;E+i\eta)|^2}\neq 0. \tag{3.19}$$

In order to do this we would require $\overline{|G|^2}\sim\eta^{-1}$. However, this can happen before or after integration, and this is how Goldstone's theorem has a chance to be avoided.

So we break down the possibilities as McKane and Stone did:

(1) $\overline{\rho(E)}\neq 0$ and $\overline{|G|^2}$ diverges after integration. By integrating over space we merely get the correlator corresponding to zero momentum, and so its reciprocal matches to the mass or self-energy. Because $\overline{|G|^2}\sim\eta^{-1}$ there are excitaitons with mass proportional to η, which tend to zero in the limit. Therefore we have massless excitations, i.e. Goldstone bosons. The presence of Goldstone modes allows for long-range correlations and so this corresponds to the "extended or conducting phase."

(2) $\overline{\rho(E)}\neq 0$ and $\overline{|G|^2}$ diverges before integration. Now $\overline{|G|^2}$ must diverge for all momenta, not just zero. Therefore there is no necessity for Goldstone bosons. If this is the case the remaining density of states corresponds to $\overline{\rho_l(E)}$ and so there must be localized states. In the absence of Goldstone bosons long-distance modes are lacking and so this is the localized phase.

(3) $\overline{\rho(E)}=0$ and the Ward identity provides no information. There is no broken symmetry, no Goldstone modes and no localization.

And so we see the second possiblity is the alternative to Goldstone's theorem, possible in the framework thanks to localization. There are a couple of assumptions in this model: there are no interactions among states, and effects are presumed to be universal so the simplest noise model should still give us the proper features. A small note: the fact that V follows a Gaussian distribution was not necessary in this argument, that enters into play in the field theoretic model where there exist a few more subtleties to this analysis.

4 Basic scaling argument

There exists a simple argument to determine a dependence of dimensionality on localization. Consider a d-dimensional system with a hypercubic lattice, which in general contains extended and localized states. We will analyze the presence of extended versus localized states through the scaling of the dimensionless conductance $g(L) = \frac{2\hbar}{e^2}G(L)$ (also known as the "Thouless number"), which will determine propagation at a length scale L and is our relevant coupling constant. $G(L)$ is the dimensionful conductance. We follow the scaling argument done by the "gang of four" [3] with additional guidance from Ref. [5].

We consider scaling L by some multiplicative factor to bL. Then we should have some scaling law

$$g(bL) = f(b, g(L)). \tag{4.1}$$

We can also represent the renormalization through a continuous scaling by the beta function defined by

$$\beta(g(L)) = \frac{d\ln g(L)}{d\ln L}. \tag{4.2}$$

The sign of the beta function will determine whether we have a conducting or localized state. If $\beta > 0$ then the conductance grows with length scale and we must have a conducting state, and if $\beta < 0$ then the conductance dies off with increasing leangth scale and we must have localization preventing transport.

There is a simple analysis that presents what form $g(L)$ should take in a state with weak disorder. The quantity that was considered by Licciardello and Thouless [2] was $g(L) = \frac{\Delta E}{dE/dN}$. Here ΔE represents the geometric mean of the fluctuations in energy levels due to replacing periodic boundary conditions on the lattice with antiperiodic boundary conditions. The reason for this is that we wish to understand the effects of the boundary conditions. Extended states should be heavily subject to this change, while localized states should be more or less independent of it. In seeing what corresponds to the extended states we can ascertain things about the conductance as those are the states that contribute to it. The other factor, dE/dN is the mean spacing of energy levels.

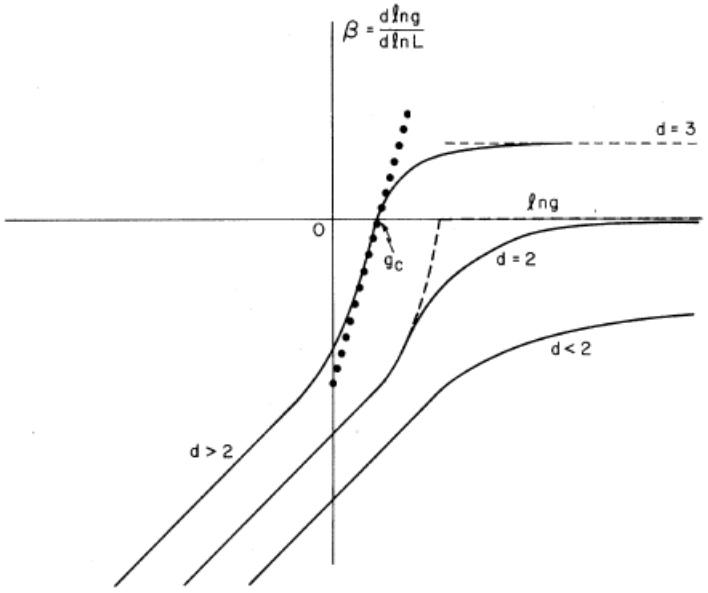
What we find is in the region where conductance is large

$$g(L) = \frac{2\hbar}{e^2}\sigma L^{d-2} \tag{4.3}$$

and hence we have the asymptotic limit

$$\lim_{g\to\infty}\beta(g) = d - 2. \tag{4.4}$$

The power of $d-2$ basically comes from a ratio of the system size (L^d) to the diffusion constant ($\sim L^2$). This corresponds to the classical limit where simple diffusion of electrons is meant to



Here is a plot of β vs. $\ln g$. We can see the smooth connection between the two asymptotic limits. The dots represent the linear expansion around the critical point. This figure is courtesy of Ref. [3].

account for conductivity.

The opposite scenario is when we have small conductance due to presence of localization, then we expect exponentially decaying correlations like

$$g(L) = g_a e^{-\alpha L}. \tag{4.5}$$

Now in this region we have

$$\beta(g) = \ln \frac{g}{g_a} \tag{4.6}$$

and by assumption we must have $\beta < 0$. It is assumed that these two regions of small and large conductance will connect smoothly under the scaling. This simple argument poses the idea that even for weak disorder there will be no long-range transport in either $d = 1$ or 2 dimensions, because for finite conductance we always have $\beta < 0$. When we move to $d \geq 3$ now we can have $\beta > 0$ and there exists some critical amount of disorder to cause localization. Intuitively, the dimensional dependence on disorder is not surprising. In higher dimensions there exist many more options for free modes to avoid stumbling blocks in their paths.

The metal-insulator transition will occur at the point $\beta = 0$ which occurs for some critical coupling g_c. Let's expand in this region. We observe a linearized renormalization formula

$$\beta(g) = \frac{g - g_c}{\nu g_c} \tag{4.7}$$

where ν is some necessarily positive number that gives the slope in this expansion (a simple

visual is provided by the dotted line in the above figure). Thereby this is an unstable fixed point, we always flow away from this state with increasing length scale. Following the derivation in Ref. [5] (4.7) approximately reads

$$\frac{1}{g_c}\frac{dg}{d\ln L} = \frac{g - g_c}{\nu g_c} \tag{4.8}$$

which is easily integrated over the length scales L_0 to L to obtain

$$\left|\frac{L}{L_0}\right| = \left|\frac{g(L) - g_c}{g(L_0) - g_c}\right|^{\nu}. \tag{4.9}$$

The convenient length definition

$$\xi = L_0 \left|\frac{g_c}{g(L_0) - g_c}\right|^{\nu} \tag{4.10}$$

allows us to write

$$g(L) = g_c \left[1 \pm \left(\frac{L}{\xi}\right)^{1/\nu}\right]. \tag{4.11}$$

The plus or minus corresponds to the conducting phase (+) or the insulating phase (−). We might associate ξ as the localization length as it diverges at the critical point (and when $L = \xi$ we have no conductance if we start in the insulating phase, although of course this functional form should be limited to near the critical point). As was necessary g decreases with increasing L in the localized phase and increases in the conducting phase.

We can also plug in our result to the usual conductivity formula, using that the bare conductivity is $\sigma = g/L^{d-2}$ then

$$\sigma \sim \xi^{2-d} \tag{4.12}$$

or in the manner the gang of four expressed it

$$\sigma \sim \left|\frac{E - E_c}{E_c}\right|^{(d-2)\nu}, \tag{4.13}$$

which will continuously tend to zero as the critical point is approached.

This model is a very famous scaling argument, particularly so it seems that the colloquial name the "gang of four" was given to the original creators. There are numerous materials that follow this sort of conductivity scaling and whose critical exponent has been measured. The exponent $s = (d-2)\nu$ has been measured for such materials as Ge:Sb (germanium crystals doped with antimony) [8]. The exponents observed depended on the doping level, and were observed for Ge:Sb to vary from $s = 0.55$ without doping to $s \approx 1$ at around 20% doping. It is generally observed that the exponent s varies from 1/2 to 1 [5]. This demonstrates a sharper decrease in conductivity as the critical point is approached for the higher doped cases, as we might logically believe as the higher doped case should correspond to more disorder and

stronger localization.

However, the scaling analysis here is actually a little naïve. It seems that weak disorder invariably leads to localization in one and two dimensions, but there are known examples of exceptions where a disordered one-dimensional system can have extended states. There are several assumptions: The system exhibits a classical conductivity form on a mascroscopic scale, determined by electron diffusion. The propagating modes need not be electrons however; in fact, localization can be avoided in a thin-film insulator-superconductor transition, where the electrons are localized but Cooper pairs are able to propagate [5]. This is a disordered two-dimensional conducting (superconducting even) state. This is a result demonstrated experimentally such as in Ref. [7], where applying a magnetic field in "atomically disordered α-InO_x" films results in such an insulator-superconductor transition.

A theoretical example of dodging the scaling law's implications is the random-dimer model [6]. Consider a one-dimensional lattice, where all the sites are identical of type a. Then, randomly, sites will be disordered which we can represent by calling such sites b. The dimer model puts a restriction on how disorder is added to the system: never is a single impurity placed, but instead impurities are placed adjacently and the impurity pair (b-b) is what is to be placed randomly. What happens is that at resonance each dimer only causes a phase shift in a propagating mode, instead of causing its dissipation, and we can have extended states. To paraphrase Phillips [5], this example leads to an insight that localization at certain energies can be avoided if the disorder contains a plane of symmetry.

5 Summary

A number of conventional phase transitions methodologies have been examined in the context of localization transitions. We have shown the concept of an order parameter and critical exponents with some rough examination near the critical point, the alternative to Goldstone's theorem due to the possiblity of localized states, and a simple renormalization procedure that indicated features to a class of materials following a metal-insulator transition and its dependence on dimensionality.

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