Phase transitions of traffic flow

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Abstract

This essay introduces a basic model for a traffic jam phase transition. Order parameter will be defined, motivated from physical reasoning. Phase diagram of the system shows there is a critical behavior, from which we can deduce critical exponents similar to statistical mechanical exponents.

I. INTRODUCTION

Traffic jam is one example of a non-statistical mechanics phenomenon that exhibits phase transition. One phase, called the free phase has all cars traveling at the maximum allowed speed. The other phase, called the jam phase occurs when the average speed of cars is less than the speed limit, that is, the flow of cars is slowed down.

In this essay, we will explore a basic model describing this phase transition. By comparing with statistical mechanics results, we will show that there is a critical behavior, and then proceed to calculate various critical exponents. Finally we discuss various improvements that can be made to the model.

II. BASIC MODEL

A basic model describing traffic jam was introduced by Nagel and Schreckenberg [1]. This model is defined on a space-time lattice. Each lattice point S_i may be empty or be occupied by a car, and after each time step, the positions $\{x_j\}$ of all cars are updated. The lattice used here is simply a one-dimensional lattice with a periodic boundary condition (a ring), so it models a single-lane traffic. Each car has a velocity v_j (with maximum velocity v_{max}) which determines how far it will move after each time step. For simplicity, the lattice spacing is assumed to be 1.

The time evolution follows these rules:

- $v_j(t+1) = v_j(t) + 1$ when the car ahead is located farther than $v_j(t) + 1$ and $v_j(t) < v_{max}$.

  A car may accelerate as long as its speed is below speed limit and the car in front is far enough.

- $v_j(t+1) = [x_{j+1}(t) - x_j(t)] - 1$ if $x_{j+1}(t) - x_j(t) \leq v_j(t)$.

  Here, $x_{j+1}(t) - x_j(t)$ is the distance to the next car. A car must decelerate if the next car is close enough.

- If $v_j(t+1) > 0$, then $v_j(t+1) \to v_j(t+1) - 1$ with a fixed probability p. This quantity p has an important role which will be discussed later.

- $x_j(t+1) = x_j(t) + v_j(t+1)$

It should be noted that these rules ensure no crashing will occur, because the location of any car at time $t+1$ is always behind the location of the next car at a *previous* time t.

If we observe how the decision to accelerate/decelerate is done in this model, we'll see that it is slightly different from real traffic. In real traffic, drivers adjust their speed according to the next car's position *and* velocity. For example, a driver may start to decelerate when the next car slows down. However, in this model, only the next car's position determines whether a car will accelerate or decelerate. Even with this simplification, this model has a phase transition.

III. PHASE TRANSITION

The two physical quantities of interest here are density of cars ρ and flux q. The density ρ is defined to be [2]

$$\rho = \langle n_i(t) \rangle \,, \tag{1}$$

where n_i is the occupation number of lattice point S_i, and the angled bracket denotes time averaging over a large time period. As pointed out by Souza and Vilar in [2], if the averaging is done over a long enough time, the density measured will be associated with the steady-state, and hence will be independent of S_i.

The flux q is defined as

$$q = \langle m_i(t) \rangle \,, \tag{2}$$

where $m_i(t)$ is the number of cars passing S_i between time t and $t+1$. Here, the flux q is also a steady-state value, independent of S_i.

Now let's think about the physical distinction between a free (no jamming) phase and a jammed phase. Suppose there are very few cars (small ρ) and for now assume $p = 0$ (no random deceleration). Intuitively, we would expect to have no jamming in the steady-state. This means every car is expected to move at the speed limit v_{max}, and hence the flux is $q = \rho v_{max}$.

As the traffic gets more cars (*i.e.* ρ increases), we expect that the steady-state velocity eventually becomes less than v_{max}. Thus, the flux $q = \rho v$ will be less than ρv_{max}. The relation $q < \rho v_{max}$ (or equivalently, $1 - q/\rho v_{max} > 0$) characterizes the jamming phase.

Motivated by this observation [2], we define the order parameter to be

$$M = 1 - \frac{q}{\rho v_{max}}, \tag{3}$$

so that $M = 0$ if $\rho \leq \rho_c$, and $M \neq 0$ otherwise.

If we compare this to a statistical mechanics system (say, nearest-neighbor Ising), the role of temperature is analogous to $1/\rho$. At high "temperature" ($\rho < \rho_c$), the system is in a disordered state ($M = 0$, free phase). Below a "critical temperature", *i.e.* $\rho > \rho_c$, the system obtains a non-zero "magnetization" M (jammed phase).

Fig. 1 shows the results of numerical simulations by Souza and Vilar [2].

IV. PHASE DIAGRAM

In Fig. 1, we can see that if we start from the free phase at $p = 0$, increasing p causes M to have a non-zero value. This is analogous to spin system, where increasing external magnetic field H from zero on a disordered phase produces a non-zero magnetization (polarizes the system). Noticing this similarity, Souza and Vilar suggested that p is the variable conjugate to M [2], in the same way that external magnetic field is conjugate to magnetization.

With this knowledge, we can immediately obtain the phase diagram in the $p-\rho$ plane (keeping v_{max} fixed), shown in Fig.2(a). Here, the symbol "2" indicates a continuous

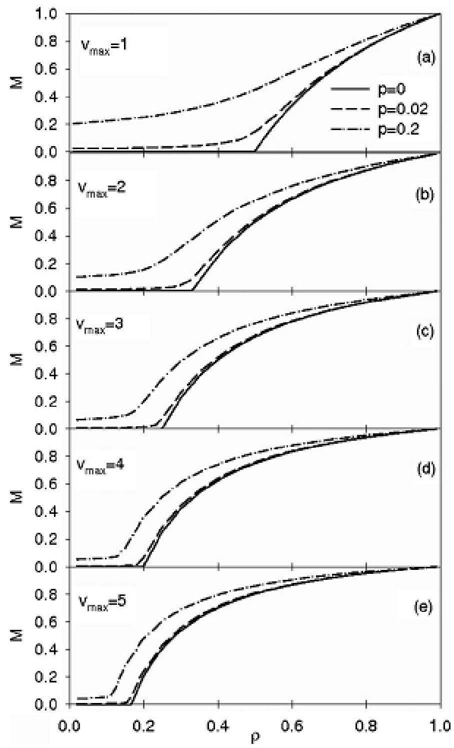

FIG. 1: Plots of order parameter M versus density ρ for various v_{max} and p. Reproduced from [2].

phase transition along the line $p = 0$ from free phase to jammed phase.

If we take $p = 0$, there is a known analytic solution [2]:

$$M = \begin{cases} 0, & \text{if } \rho \leq \rho_c \\ \frac{1}{v_{max}} \frac{\rho - \rho_c}{\rho \rho_c}, & \text{if } \rho > \rho_c \end{cases} \tag{4}$$

and $\rho_c = 1/(1 + v_{max})$. The resulting phase diagram in $\rho - 1/v_{max}$ plane is shown in Fig.2(b).

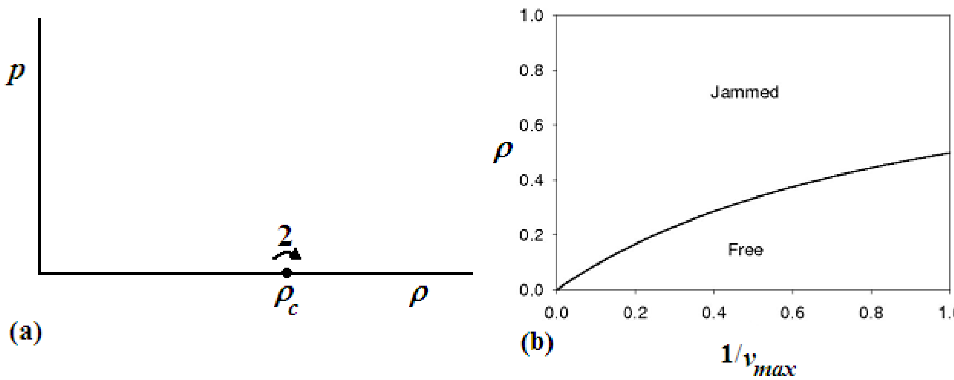

FIG. 2: Phase diagrams for (a) p and ρ plane (v_{max} fixed), (b) ρ and $1/v_{max}$ plane ($p = 0$). Part (b) has been reproduced from [2].

V. CRITICAL EXPONENTS

In the previous section, we had assumed that p is conjugate to M. Now, a susceptibility can be naturally defined as

$$\chi = \left.\frac{\partial M}{\partial p}\right|_{p=0} . \tag{5}$$

From Fig.2(a), we see that there is a critical point at $p = 0$ and $\rho = \rho_c$. Thus, we may define critical exponents β, γ, and δ as follows [2] (there is no α in this case since there is no clear analog of heat capacity):

- At $p = 0$, for $\rho > \rho_c$, but small $(\rho - \rho_c)/\rho_c$, we have $M \sim (\rho - \rho_c)^\beta$ and $\chi \sim (\rho - \rho_c)^{-\gamma}$
- At $\rho = \rho_c$ (critical "isotherm"), we have $M \sim p^{1/\delta}$

Souza and Vilar calculated these exponents numerically [2] for several values of v_{max} with the results shown in Table I. All of these results are consistent with the well-known algebraic relation $\beta\delta = \beta + \gamma$.

What about the correlation length? Roters, et. al. [4] analyzed the correlation length of the velocity (that is, a measure of how correlated the velocity of a group of cars close

TABLE I: Critical exponents by Souza and Vilar [2]

v_{max}	β	δ	γ
1	1	2	1
2	1	1.73	0.73
3	1	1.61	0.60
4	1	1.54	0.54
5	1	1.48	0.47
∞	1	1	0

to each other) using the so-called dynamical structure factor

$$S(k,\omega) = \frac{1}{NT}\left\langle \left| \sum_{n,t} v_n(t) e^{i(kn-\omega t)} \right|^2 \right\rangle . \tag{6}$$

This is essentially the Fourier transform of the two-point function, and hence the correlation length can be read off from the asymptotic behavior of $S(k,\omega)$.

Using this technique, they were able to show [4] that near $\rho = \rho_c$,

$$\xi \sim (\rho - \rho_c)^{-\nu}, \tag{7}$$

with $\nu = 0.92 \pm 0.05$.

VI. FURTHER IMPROVEMENTS

There are several ways to improve this very basic model. One way is to include the possibility of having more than one lane, taking into account the fact that cars may switch lanes [6].

A continuum theory (assuming large clusters of jammed phase) has also been analyzed in the literature [5]. This theory uses partial differential equations instead of a finite time step evolution. Finally, [3] explores the mean field approximation of the model discussed in this essay.

VII. CONCLUSION

We were able to show using a simple discrete space-time model, that traffic flow exhibits a phase transition from a free phase to a jammed phase. By making analogies to spin system, we managed to construct a phase diagram and also showed that there is a critical point. The computation of the critical exponents further demonstrated the universality of the critical behavior. This is a surprising result because the system is not a statistical mechanics system, and it shows that the knowledge of critical phenomenon is very useful in many other research areas.

[1] K. Nagel and M. Schreckenberg, J. Phys. I (France) **2**, 2221 (1992).

[2] A.M.C. Souza and L.C.Q. Vilar, Phys. Rev. E **80**, 021105 (2009).

[3] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Phys. Rev. E **51**, 2939 (1995).

[4] L. Roters, S. Lübeck, and K.D. Usadel, Phys. Rev. E **59**, 2672 (1999).

[5] B.S. Kerner, and S.L. Klenov, Phys. Rev. E **56**, 4200 (1997).

[6] B.S. Kerner, S.L. Klenov, and P. Konhäuser, Phys. Rev. E **80**, 056101 (2009).