Critical Phenomena in Gravitational Collapse

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I briefly review the critical phenomena in gravitational collapse with emphases on connections to critical phase transitions.

1 Introduction

It was well known that strong enough initial configurations of matter field would result in gravitational collapse to black holes, whereas weak enough initial data become dispersed to infinity. However, what happens in the intermediate region was not clear until the discovery of the critical phenomena in gravitational collapse by Choptuik in 1993 [1]. He examined numerically the spacetime evolution of massless scalar field minimally coupled to gravitational field. The model studied is the simplest one so that numerical studies are practical and accurate enough.

His discovery is remarkable. There's three major phenomena. The first is the mass scaling law in the resulting black hole mass M:

$$M \simeq k(p - p_*)^\gamma \tag{1}$$

p is parameter in a one-parameter family of initial data which is varied to give different initial conditions. While constant k and critical value p_* depend on the particular one-parameter family, exponent γ is universal. The second finding is universality. For a finite time in a finite region of space, the spacetime generated by all initial data takes the same form as long as they are close to the so called critical conditions. The commonly approached solution is called the critical solution, and it has an amazing property, the third phenomenon, called scale-echoing. In the model studied, the critical solution is scale invariant by a factor e^{Δ}:

$$\phi_*(r,t) = \phi_*(e^{\Delta}r, e^{\Delta}t) \tag{2}$$

where $\Delta \simeq 3.44$.

Following his work, a lot of other matter models were studied, and similar critical phenomena were discovered. It is now clear that the critical phenomena in gravitational collapse are common features in many gravitational fields. The findings can be summarized as follows:

1. Δ and γ are universal within a given field, but can be different for different fields.

2. There are actually two types of critical phenomena: type I and type II, named after analogy to critical phase transitions in statistical mechanics. The type of critical phenomena is related to symmetry properties of the critical solution. A critical solution with scale invariance (either discontinuous self-similarity (DSS) or continuous self-similarity (CSS)) is related to type II critical phenomena, such as the case studied by Choptuik [1], where the 'order parameter' M is turned on continuously at criticality. A critical solution with time periodicity instead of self-similarity or scale invariance is related to type I phenomena where M is always finite when a black hole is formed.

The research in this area involves many technical details. For the purpose of this review, I'll only focus on the aspects related to critical phase transitions, leaving out most details. I'll mainly follow the review work by Gundlach [3] and Gunlach and Martin-Carcia [2], where one can find a quite complete description of the area.

2 Numerical studies

Due to the complexity of Einstein's field equations, most progress in the area is achieved by numerical simulations. Even for computer simulations, most work is done in the simplest cases: in most cases, spacetime with spherical symmetry; in a few cases, spacetime with axisymmetry.

2.1 Choptuik's study on spherically symmetric massless scalar field

The governing equations are Einstein's equations:

$$G_{ab} = 8\pi(\nabla_a \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \nabla^c \phi) \tag{3}$$

and scalar field equation:

$$\nabla_a \nabla^a \phi = 0 \tag{4}$$

The metric is chosen to be:

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2 d\Omega^2 \tag{5}$$

Introducing the auxiliary variables

$$\Phi = \phi_{,r}, \Pi = \frac{a}{\alpha}\phi_{,t} \tag{6}$$

the wave equation (4) becomes:

$$\Phi_{,t} = (\frac{a}{\alpha}\Pi)_{,r} \tag{7}$$

$$\Pi_{,t} = \frac{1}{r^2}(r^2 \frac{\alpha}{a}\Phi)_{,r} \tag{8}$$

combined with the two Einstein equations:

$$\frac{a_{,r}}{a} + \frac{a^2 - 1}{2r} - 2\pi r(\Pi^2 + \Phi^2) = 0 \tag{9}$$

$$\frac{\alpha_{,r}}{\alpha} - \frac{a_{,r}}{a} - \frac{a^2 - 1}{r} = 0 \tag{10}$$

gives the equations that were numerically solved by Choptuik using finite-difference techniques and an adaptive mesh-refinement algorithm. Choptuik focused on one-parameter families of solutions by fixing the initial scalar field configurations and varying only one variable in the initial data. For example, one of the forms he studied is $\phi = \phi_0 r^3 exp(-[(r - r_0)/\delta]^q)$. The one-parameter family can be from any one of the parameters ϕ_0, r_0, δ, q. They control the strength of the initial field.

What he found is that for $p > p_*$ corresponding to strong initial field, the solution collapsed to a black hole after a long time; whereas for $p < p_*$, the scalar field dispersed to infinity and left over a flat spacetime. For initial data close to critical parameter, all solutions spend a long time evolving very close to the critical solution. Finally, some go to gravitational collapse to black holes and others disperse away. There's nothing special about the critical point in that one can't tell which near-critical solution would become black hole or disperse to infinity. One has to fine-tune the parameter and wait for long enough time to locate the critical point.

2.2 Studies on other matter fields

Numerical studies on many matter fields have been carried and similar critical phenomena are observed. Those fields include perfect fluid with pressure proportional density, massive scalar field, massless scalar electrodynamics, massive vector field, sigma model, SU(2) Yang-Mills, axisymmetric gravitational waves, etc. Some fields show type II critical phenomena, while others show type I critical phenomena, and some show a mix of type I and II phenomena in different regions.

Most studies are restricted to spherical symmetry for numerical simplicity. However, it's important to know if the critical phenomena are generic in more realistic fields, for example, fields with non-spherical symmetry and associated angular momentum. There's some work on non-spherically perturbations around spherically critical collapse using linear perturbation analysis. But there's still few numerical studies to verify the predictions. Recently, there's a numerical work by Olabarrieta et al. [5] on gravitational collapse of a scalar field with angular momentum, but still in spherical symmetry. The work is not directly related to realistic situation. Nevertheless, their results are interesting and may shed some light on the critical collapse with angular momentum. They found both echoing exponents Δ_l and mass-scaling exponents γ_l decrease rapidly with increasing l, the eigenvalue of angular momentum. They argued that the stability of critical solution increases with increasing angular momentum, which is believed to be due to the momentum potential barrier that helps stabilize the collapse to black hole formation.

3 Discussions on critical phenomena

3.1 Phase space picture

It is agreed that critical phenomena in gravitational collapse can be described by ideas from dynamical systems. The universality of the critical solution and critical exponent can be understood in terms of perturbations of critical solutions. We can treat GR as an infinite-dimensional dynamical system. The phase space is the set of all possible initial data. The numerical evidence so far for one-parameter families suggests that the critical solution lies in a hypersurface embedded in the phase space of smooth, asymptotically flat initial data . The hypersurface is the basin of attraction of the critical fixed point of codimension one. A one-parameter family that depends smoothly on its parameter only crosses the hypersurface once. Any initial point that starts in the critical surface never leaves it and would eventually evolve to the critical fixed point. Thus the critical surface divides the phase space into two parts. On one side, there's a black hole fixed point and on the other, a flat space fixed point. Initial parameter $p = p_*$ corresponds to a point in the critical surface. Initial data with $p > p_*$ would eventually be attracted to the black hole fixed point and data with $p < p_*$ would evolve into the flat space fixed point.

Figure 1 illustrates the time evolution of spacetime solutions. For any trajectory starting near the critical surface, it evolves almost parallel to the surface towards the critical point. As it approaches the critical point, it slows down and spend some time near the critical point. Eventually, it moves away from the critical point towards an attracting fixed point depending on which side of the critical surface it starts from.

This picture explains the origin of the universality. Any initial data set that is close to the critical surface spends quite long time near the critical surface and the critical point. When it finally evolves to the fixed point, it appears to come from the critical point. All information in the initial data is washed away during the evolution expect the initial distance, i.e. $|p - p_*|$ from the critical surface. This picture is very similar to the renormalization group (RG) flow picture and implies connection to the RG transformation, which is discussed briefly later. It's worth noticing that the attractor in the critical surface can also be a limit cycle corresponding to DSS. The qualitative description is not affected by this difference.

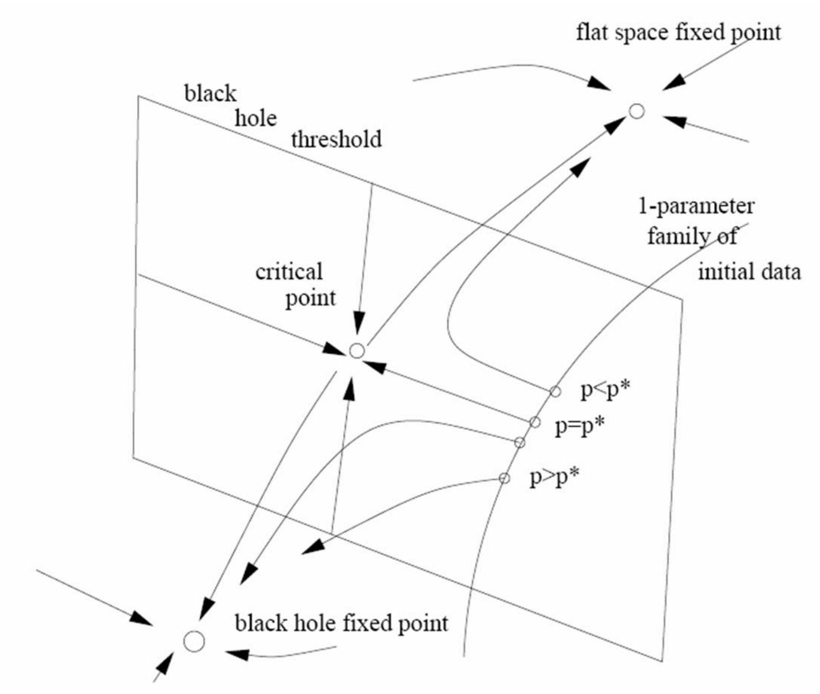

Figure 1: The phase space picture for the critical collapse. The arrow lines are time evolutions of spacetimes. The line without an arrow is a one-parameter family of initial data. Figure from Fig.1 of [3]

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3.2 Linear perturbation of critical solution

Let's now see how to quantitatively understand the critical phenomena, in particular, calculate the critical exponent.

First consider type I critical phenomena. In type I critical phenomena, the critical solution has time symmetry. For simplicity, consider a time independent critical solution. In the case of spherical symmetry with time coordinate t and radial coordinate r, it can be written as:

$$Z_*(t,r) = Z_*(r) \tag{11}$$

Linear perturbing the critical solution and keeping the only unstable mode (there's only one growing mode), we get the evolution of near-critical solution:

$$Z(r,t) \simeq Z_*(r) + \frac{dC_0}{dp}(p_*)(p-p_*)e^{\lambda_0 t} Z_0(r) \tag{12}$$

After long enough time, all the decaying modes die out. Define a period t_p during which the solution stays in the near-critical region:

$$\frac{dC_0}{dp}|p-p_*|e^{\lambda_0 t_p} \equiv \epsilon \tag{13}$$

where ϵ is an arbitrary small positive constant. The solution at $t = t_p$ is independent of $|p-p_*|$, and so the final black hole mass if formed is expected to be fixed, independent of $|p-p_*|$. This can be understood according to the picture described above: all trajectories that go into black hole fixed point appear to come from the same point given by equation (12) at $t = t_p$. The time scale during which the near-critical solution is close to critical solution is:

$$t_p = -\frac{1}{\lambda_0} log|p-p_*| + const. \tag{14}$$

This sets a mass scale in the problem. Hence, by dimension argument, it's reasonable to expect the black hole mass to turn on discontinuously from zero with the magnitude determined by the mass scale in the problem. By contrast, type II phenomena, which are scale-invariant, occur in situations where either there's no scale in the field equations or the scale is dynamically irrelevant. Actually, in some cases where the importance of the scale varies in different regions of initial data, one sees both types of phenomena.

Now let's turn to the type II critical phenomena. It's more convenient to change coordinates to

$$x = -\frac{x}{t}, \tau = -log(-\frac{t}{l}), t < 0. \tag{15}$$

The critical solution has the property, in terms of new coordinates

$$Z_*(x, \tau + \triangle) = Z_*(x, \tau) \tag{16}$$

In the case of CSS, the critical solution is independent of τ. Let's consider the case of CSS first. Similar to the procedure in the discussion of type I phenomena, one has

$$Z(x,\tau) \simeq Z_*(x) + \frac{dC_0}{dp}(p-p_*)e^{\lambda_0 \tau} Z_0(x) \tag{17}$$

for τ long enough. By defining the similar period as in type I discussion, τ_p, we have

$$\tau_p = -\frac{1}{\lambda_0} log|p - p_*| + const. \tag{18}$$

Again, we have the near critical solution at $\tau = \tau_p$:

$$Z(x, \tau_p) \simeq Z_*(x) \pm \epsilon Z_0(x) \tag{19}$$

where the $\pm$ sign is the sign of $p - p_*$. In the original coordinates, the solution looks like

$$Z(r, 0) \simeq Z_*(-\frac{r}{L_p}) \pm \epsilon Z_0(-\frac{r}{L_p}), L_p \equiv Le^{-\tau_p} \tag{20}$$

The intermediate data given by equation (20) depend on the initial data only through L_p. The field equations do not have an intrinsic scale. Since L_p is the only scale in the solution, which has the same unit as mass, we must have

$$M \sim L_p \sim (p - p_*)^{\frac{1}{\lambda_0}} \tag{21}$$

One can easily read off the critical exponent $\gamma = 1/\lambda_0$. In the case of DSS, the scaling law needs to be modified. A 'fine structure' of small amplitude is superimposed:

$$logM = \gamma log(p - p_*) + c + f(\gamma log(p - p_*) + c) \tag{22}$$

with $f(z) = f(z + \triangle)$. The periodic function f is again universal and only c depends on initial data.

3.3 Analogy to critical phase transitions and renormalization group

The main features of critical phenomena discussed above are identified in critical phase transitions in statistical mechanics. In particular, one noticed that type II phenomena occur with scaling law and critical exponent when the critical solution has scale invariance. This is reminiscent of second order phase transition in thermodynamical systems, such as liquid-gas transition and spontaneous magnetization in a ferromagnetic material. In both systems, at critical point, the correlation length diverges and physics in length scales on microscopic levels, such as the atomic scale, become irrelevant at the critical point. The scale invariance, that physics looks the same at different scales, is the origin of the universality of scaling behaviours in critical phase transitions.

To calculate thermodynamical properties near critical point, one can coarse-grain original physical systems and construct renormalization group flows. The renormalization group can be considered as a dynamical system and hence is analogous to the phase space picture in gravitational collapse. Consider the ferromagnetic material in the absence of an external magnetic field. The critical surface in this case is a hypersurface of codimension one .(A point in this case with one dimension less than a line of temperature T-axis). The renormalization group flows in this scenario is almost the same as the time-evolution of one-parameter families in the phase space in critical collapse. The calculation of mass scaling law of type II phenomena above is just the calculation of critical exponent of correlation length. One then identifies black hole mass M as correlation length ξ, and $p - p_*$ as $T_* - T$. The only difference here is that the RG flow goes towards larger scales and time evolution is towards smaller scales. So ξ diverges, while M vanishes at the critical point. It's interesting to note that if one brings in the external magnetic field B, then in some cases, the angular momentum L of the initial data in gravitational collapse is the equivalent of B. The final angular momentum of the black hole is the equivant of magnetization M. The analogy of type II phenomena with critical phase transition is extended to two parameters.

Due to the close connection of RG to the critical phenomena in gravitational collapse, it's natural to expect that the theory of RG may be used to study the gravitational critical collapse. Hara et al. [4] carried a systematic RG study on perfect fluids and obtained accurate critical exponents in agreement with numerical studies. However, to achieve a full RG in arbitrary GR spacetimes is highly non-trivial. There has been little progress in this direction.

3.4 Observations

Critical phenomena in gravitational collapse has applications to primordial black hole formation. Results from analysis on critical gravitational collapse may be compared to observations. Green and Liddle [6] compared results from perturbation analysis by Niemeyer and Jedamzik to MACHO observations. Figure 2 shows the Niemeyer and Jedamzik PBH mass distribution and mass distribution of the MACHOs from microlensing observations. The PBH mass distribution is considerably broader than the observations. The discrepancy may be due to lack of observation data. Further microlensing searches may fit the broad spread of the PBH mass distribution.

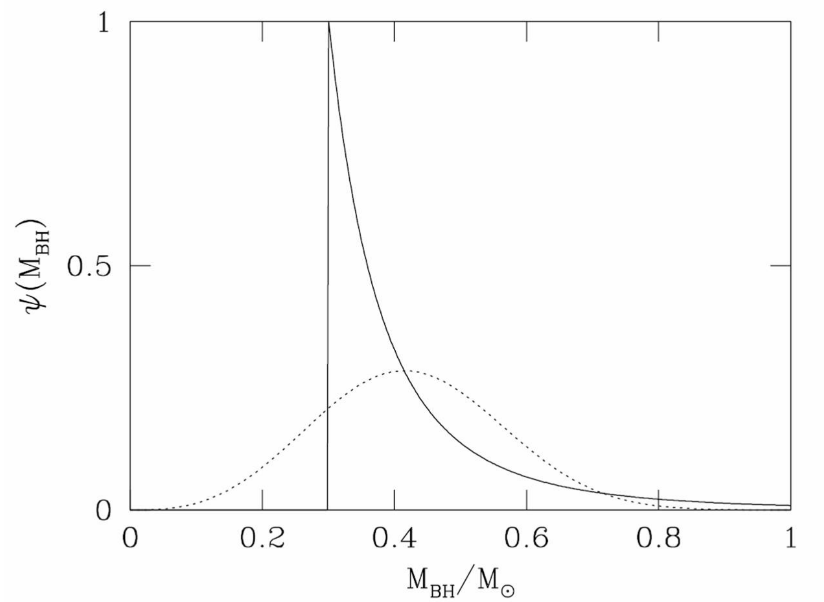

Figure 2: The solid line is the observed mass distribution of the MACHOs, and the dashed line is the Niemeyer and Jedamzik mass function. Figure from Fig.7 of [6]

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4 Summary

Critical phenomena in gravitational collapse is a very rich area. Due to the complexity of the Einstein's field equations, most progress is made by numerical simulations of rather simplified models. On the other hand, ideas from phase transition can still shed some light on understanding physics pictures underlying the phenomena. With further investigations on more general matter models, on quantum effects on gravitational collapse, and on many other issues, the analogy to phase transitions in statistical mechanics would continue to help understand the physics.

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