Deconfinement Phase Transition in QCD

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At low energy, the interactions between quarks are so strong that they have to form a bound state because of color confinement. However, at high energy, due to asymptotic freedom, the effective coupling becomes small. Quarks will be deconfined, leading to a state of quark-gluon plasma. This is the so-called deconfinement phase transition. Starting with some background, we will describe the basic physics behind this phase transition. Then lattice QCD is introduced to give us more quantitative results. Possible connections with condensed matter physics will also be considered.

I. INTRODUCTION

A. Formulations of QCD

Quantum Chromodynamics, or QCD for short, is the theory of the strong interactions. It is a non-Abelian gauge field theory with the Lagrangian

$$L = -\frac{1}{4}\mathrm{tr}F^{\mu\nu}F_{\mu\nu} + \sum_{j=1}^{f}\bar{\psi}_j\left(i\not{D} - m_j\right)\psi_j \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu + ig\left[A_\mu, A_\nu\right] \tag{2}$$

and the covariant derivative

$$D_\mu = \partial_\mu + igA_\mu \tag{3}$$

For those who are familiar with QED, QCD does have the same prototype. ψ_j is the spin-1/2 fermion field of quarks while A_μ is the bosonic field of gluons, just like those of electrons and photons in QED.

In fact, QCD can be constructed by the same fundamental symmetry principle underlying QED. As we know, QED is invariant under the local gauge transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \tag{4}$$

However, the derivative $\partial_\mu\psi(x)$ does not follow the simple transformation law since $\psi(x)$ and $\psi(x+\epsilon n)$ transform independently under the symmetry Eq. (4). The solution is to define a quantity called comparator $U(y,x)$, which transforms as

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \tag{5}$$

In this way, $\psi(y)$ and $U(y,x)\psi(x)$ will have the same transformation. And the so-called covariant derivative is defined as

$$n^\mu D_\mu\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right) \tag{6}$$

where $U(x+\epsilon n, x)$ can be expanded as

$$U(x+\epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) \tag{7}$$

Thus the covariant derivative takes the form as Eq. (3) and the gauge field A_μ transforms as

$$A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x) \tag{8}$$

Note that here we have extracted a coupling constant g, which would depend on the normalization of ψ and A_μ. In QED, it is just the electron charge e.

These terms involving only A_μ should also be gauge invariant, which can be constructed as follows. In Fig. 1, considering the comparators along the sides of the plaquette, in the

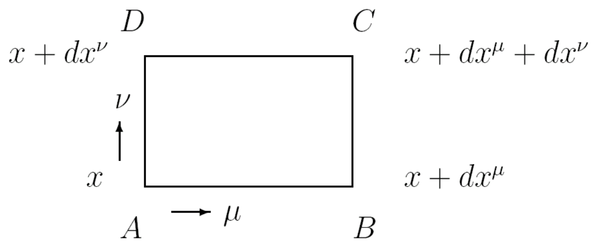

FIG. 1: Gauge invariant loop around a plaquette

direction of $A \to B \to C \to D \to A$, define the product of four comparators as

$$\mathbb{U}(x) = U(x, x+dx^\nu)U(x+dx^\nu, x+dx^\mu+dx^\nu)U(x+dx^\mu+dx^\nu, x+dx^\mu)U(x+dx^\mu, x) \quad (9)$$

From the definition of $U(x,y)$, $\mathbb{U}(x)$ should be gauge invariant. Expanding $\mathbb{U}(x)$

$$\mathbb{U}(x) = 1 - igdx^\mu dx^\nu \left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) \quad (10)$$

Thus the quantity

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (11)$$

is invariant under the gauge transformation.

So far, by the requirement of local gauge invariance, we have retrieved all the terms appearing in the Lagrangian of Eq. (1). In fact, all the other combinations of these gauge invariant terms are prevented, either by the renormalizability of the theory in $4d$, or by global, parity and time-reversal symmetry.

The gauge symmetry we have in QED is called $U(1)$ symmetry, namely a phase rotation of $\alpha(x)$. More generally we can extend $\psi(x)$ as an n-plet. And the proper gauge transformation would be

$$\psi(x) \to V(x)\psi(x) \quad (12)$$

where V(x) is an $n \times n$ unitary matrix. We can easily repeat the above symmetry arguments and reconstruct the Lagrangian as Eq. (1), except that $F_{\mu\nu}$ would be defined as Eq. (2), instead of Eq. (11). The reason is that the gauge boson field A_μ, as an matrix here, will not commute with each other. Actually, the product of comparators around the edges of a plaquette is a visualization of the commutator of covariant derivatives

$$[D_\mu, D_\nu] = igF_{\mu\nu} \quad (13)$$

which is identical with the definition of Eq. (2).

Particularly, QCD is the one with the local gauge group $SU(3)$. So ψ has three components, corresponding to the three colors of quarks. The gauge bosons will have eight varieties, corresponding with eight independent generators of $SU(3)$ Lie algebra. Moreover, we will set $f = 6$, reflecting the fact the quarks come up with six flavors: down d, up u, strange s, charm c, bottom b and top t. Quarks carry fractional charges of $+\frac{2}{3}e$ or $-\frac{1}{3}e$, and different masses. They are the elementary particles building up nucleons. For example, a proton can be viewed as a bound state of uud.

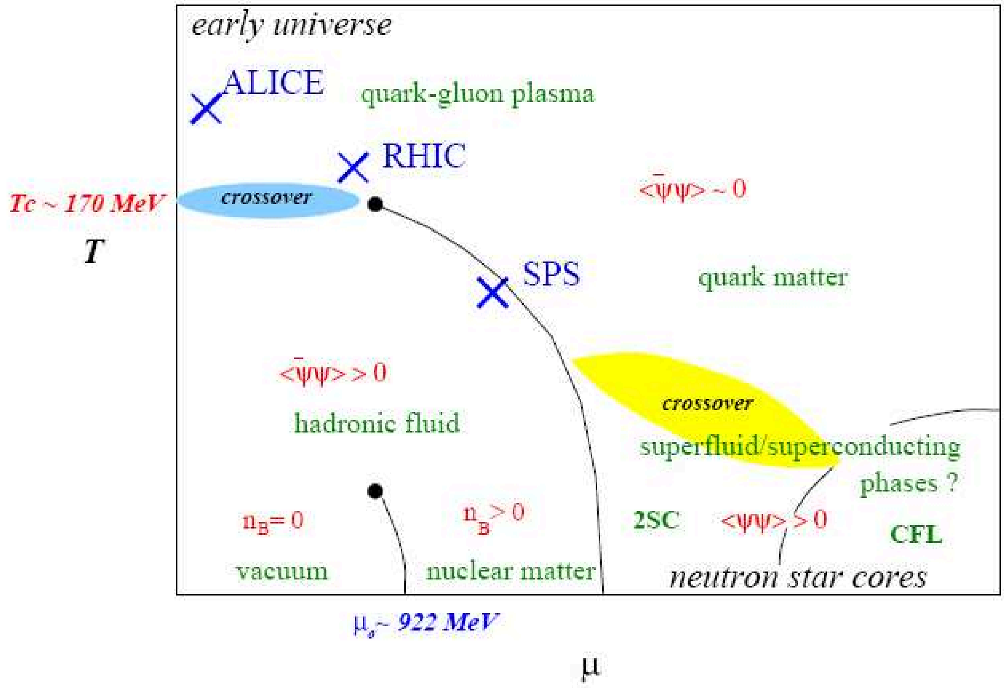

FIG. 2: Proposed phase diagram of QCD. The names of different phases are in green. Phase boundaries appear as solid lines while critical points are black dots. ALICE, RHIC and SPS are names of heavy-ion collision experiments. 2SC is where only two lightest flavors u and d form the superconductive condensate. CFL stands for color-flavor-locked where strange quarks come to the pairing, leading to a more symmetric condensate.

B. Proposed phase diagram

Although quarks are the basic constituents of QCD, isolated quarks have never been seen in nature or detected by any experiments. The reason is that at low energies, quarks prefer to form color neutral bound states called hadrons, either in the form of three-quark qqq states called baryons, such as protons (uud) and neutrons (udd), or quark - anti-quark $q\bar{q}$ states called meson, like pions (π^0:$d\bar{d}/u\bar{u}$, π^+:$u\bar{d}$, π^-:$d\bar{u}$). This phenomenon is called *color confinement*. However, as we increase the temperature, the bound states will eventually break up. Quarks and gluons become weakly coupled due to *asymptotic freedom*, forming a state called *quark-gluon plasma* (QGP). So there must be a phase transition between these two states.

From QCD Lagrangian Eq. (1), baryons qqq can only be created and annihilated in pair with anti-baryons $\bar{q}\bar{q}\bar{q}$. So the baryon number N_B, which is number of baryons minus the number of anti-baryons, is a conserved quantum number. For a system in which N_B is allowed to vary, we can define a Lagrange multiplier, chemical potential μ, as in the grand canonical ensemble. For large baryon number, where μ is large, the effective degree of freedom will be quarks, instead of hadrons. As we know, quarks are spin-1/2 fermions obeying Pauli exclusion principle. At $T = 0$, these quarks will occupy a Fermi sea, just like the electrons, forming a relativistic Fermi liquid. Moreover, the interaction due to single gluon exchange between two quarks around the Fermi surface is effectively attractive. So we will have an analogy of BCS instability, resulting in a ground state with non-vanishing $\langle qq \rangle$. The gluons will acquire mass due to gauge symmetry breaking, leading to the phenomenon of *color superconductivity.*

Following the above arguments, we can propose a phase diagram as Fig. 2. The scope of this essay will be on the $\mu = 0$ axis, the deconfinement phase transition. In Sec. II, we start with a brief description of two phases: color confinement and quark-gluon plasma. Then we move to the discussion of chiral symmetry breaking and formulate a simple physics picture of deconfinement phase transition. Sec. III is devoted to the lattice QCD. As a non-perturbative numerical treatment, it gives us a more quantitative view of the phase transition. Possible relationship with condensed matter physics will be discussed in the end.

II. DECONFINEMENT PHASE TRANSITION - A PHYSICS PICTURE

A. Color confinement

According to the renormalization group, the effective coupling g depends on the scale at which it is defined. Assuming g is small, we can do the standard perturbative renormalization and find out the running coupling constant follows

$$\frac{dg(\epsilon)}{d\ln\epsilon} = \beta_0 g^3 + \beta_1 g^5 + \cdots \tag{14}$$

where

$$\beta_0 = -\frac{1}{16\pi^2}\left(11 - \frac{2}{3}f\right) \tag{15}$$

and

$$\beta_1 = \left(\frac{1}{16\pi^2}\right)^2 \left(102 - \frac{38}{3}f\right) \tag{16}$$

Here we have set the quark mass to zero. We notice that, when $f \leq 16$ (In reality, $f = 6$), the coupling g decreases to zero as the energy scale where it is measured increases to infinity. It predicts the interaction between quarks becomes weaker as two quarks get closer. In the limit of $r \to 0$, the quarks are noninteracting. This is the so-called asymptotic freedom, which is quite counter-intuitive. Asymptotic freedom is a general feature of non-Abelian gauge theories, and by so far restricted to these theories.

However, in real world, isolated quarks are never observed. All we have are color neutral hadrons. So the interactions are actually strong here. In this regime of large g, the traditional perturbative techniques, like the Feynman diagrams summation, break down. In this sense, QCD is often referred to as non-perturbative. So how do we describe color confinement?

The simplest argument is based directly on the Lagrangian of Eq. (1). We may re-define the gauge field as $A_\mu = gA_\mu$. By this definition, $1/g^2$ indicates the energy cost to generate curvatures in gauge field. Letting g goes to infinite, $1/g^2$ would go to zero. By varying the action with respect to A_μ, we find out that the color current vanishes. Thus its zero component, color density is zero, which means color is confined.

More quantitative results come from the lattice QCD simulations, which we will introduce

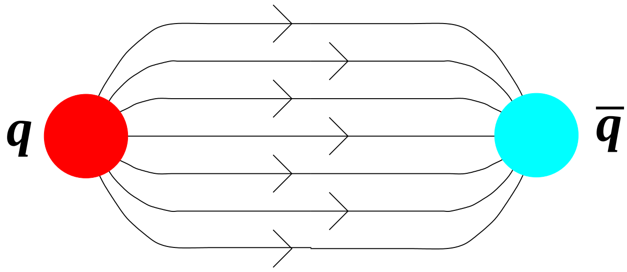

FIG. 3: gluon flux tube between a $q\bar{q}$ pair

in Sec. III. The potential between a $q\bar{q}$ pair is

$$V(r) = -\frac{A(r)}{r} + Kr \tag{17}$$

where $A(r) \propto 1/\ln(r^{-1})$, agreeing with asymptotic freedom. At large distances, the second term will take control, resulting in a linear potential between quark and anti-quark. It can be visualized by chromoelectric flux tube, as shown in Fig. 3. The gluon fields are restricted inside the tube, with a fixed diameter and constant energy density. So we are unable to separate two color sources as energy cost grows proportionally with the distance.

It should be noticed that an exact description of color confinement mechanism, based on non-perturbative approach, is still under debate. Although it is an interesting topic filled with a lot of fancy theories, we will not talk about it here.

B. Quark-gluon plasma

As temperature increases, the energy scale of gluon exchange between quarks will also increase. Asymptotic freedom predicts that the coupling between quarks and gluons will approach zero, leading to the formation of quark-gluon plasma (QGP). The temperature scale is of the order $10^{12}K$, which is achieved between $10^{-5}s$ and $10^{-4}s$ after the Big Bang. So the state of QGP does exist in the early universe. Quite amazingly, in today's large accelerators, QGP can be created by the heavy-ion collision.

The natural question to ask here is how to determine the formation of QGP. We are unable to isolate individual quarks. All we can detect are reminiscence of QGP. One of the possible signals is the so-called J/ψ suppression. J/ψ is a meson formed by charm quarks $c\bar{c}$, with the potential of Eq. (17). With the formation of QGP, the theory becomes weakly coupled and the potential takes the form of screening Coulomb potential

$$V(r) = -\frac{C}{r} e^{-r/\lambda_D} \tag{18}$$

where λ_D is the screening length. So if λ_D is smaller than the Bohr radius of J/ψ, the pair of charm quarks will break up. These charm quarks are more likely to form open pairs ($c\bar{u}$, $c\bar{d}$), leading to a suppression of the number of J/ψ, compared with that in pp collision of a similar energy. Fig. 4 shows J/ψ suppression performed at SPS. Recent experiments at RHIC and future ones at ALICE are able to probe states much farther away from phase boundaries and provide more information about QGP.

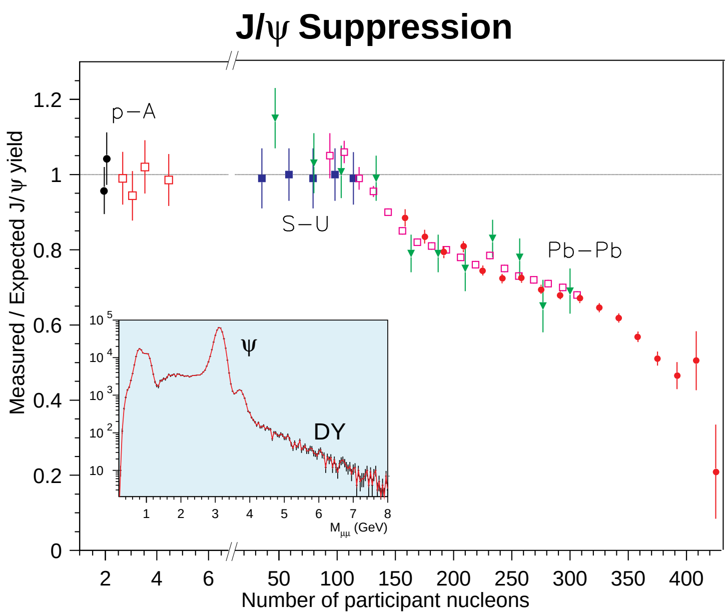

FIG. 4: J/ψ suppression at SPS. The inset shows the number of muon produced at different energies. The peak around 3GeV results from the decay from J/ψ

C. Chiral symmetry breaking

At this point, we may ask what is the order of this phase transition. Lattice calculations indicate a weak first order transition. But it may be an artifact of simulations, such as the finite size of the system. A more sensible question to ask here is what is the underlying symmetry and associated order parameter distinguishing different phases.

Let us consider a particle with spin $\vec{s}$ propagating with the momentum $\vec{k}$. Define a quantity called helicity as the projection of spin along the direction of momentum $h = \vec{s} \cdot \vec{k}/|k|$. For spin-1/2 particles like quarks, there are two eigenstates with eigenvalues $h = \pm 1/2$, usually referred to as left-handed states and right-handed states. The helicity of a quark will not be reversed by the coupling with gluons. The QCD Lagrangian does preserve the number of left-handed and right-handed quarks. This symmetry is the so-called chiral symmetry. However, unless the quark has zero mass, we can always find a Lorentz transformation to change the sign of helicity. In this sense, we say the chiral symmetry is broken by the quark mass.

If we set the quark mass equals zero, will the chiral symmetry be present in the QCD vacuum or the ground state? In fact, the simple vacuum, which are empty of particles, is not the ground state of QCD at $T = 0$ and $\mu = 0$. Due to pairing instability between quark and anti-quark, the true ground state is a condensate of bound pairs $q\bar{q}$, as illustrated in Fig. 5. This ground state does not preserve the number of left-handed and right-handed quarks, namely

$$\langle \bar{\psi}\psi \rangle \equiv \langle 0|\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L|0\rangle \neq 0 \tag{19}$$

which is a definition of order parameter. Due to the breaking of chiral symmetry, quark will acquire a constituent mass Σ, as contrast with its intrinsic mass m. Actually Σ is much

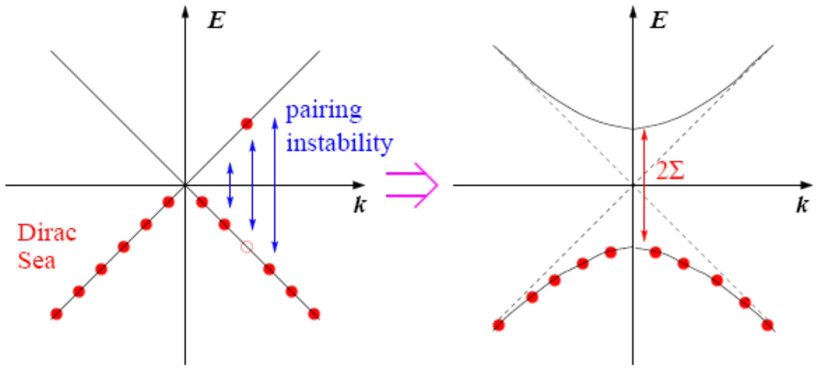

FIG. 5: Chiral symmetry breaking

larger than m. For example, the constituent mass of down quark is around 350 MeV while its intrinsic mass is around 7 MeV.

A good analogy can be made with the more familiar ferromagnetic Ising model. $\langle\bar{\psi}\psi\rangle$ plays the role of magnetization M, while the intrinsic mass m is just like external field h. When quark has a mass of zero, $m = 0$, the deconfinement phase transition is of second order, signified by the breaking of chiral symmetry, just as in the Ising model under zero external field, we go from paramagnetism to ferromagnetism by breaking the Z_2 symmetry. When m is non-zero, which is the realistic case, the behavior turns out to be a crossover, rather than a true phase transition, just like that we encountered in Ising model with $h \neq 0$. That is why we have a crossover close to $\mu = 0$ in the phase diagram Fig. 2.

III. DECONFINEMENT PHASE TRANSITION - MORE QUANTITATIVE

A. Introduction to lattice QCD

As we mentioned before, QCD is non-perturbative in the strong coupling limit. Lattice QCD discretizes our space-time and does the path integral numerically. It turns out to be a very powerful tool for studying the deconfinement phase transition.

We start with the partition function or the generating functional in Euclidean space-time

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left(-\int d^4x\mathcal{L}\right) \tag{20}$$

By dividing the space-time into a grid of size $N_T \times N_S \times N_S \times N_S$, the above functional integral can be evaluated by varying functions on each lattice.

However, there is a very important subtlety here, namely the gauge invariance. As shown at the very beginning of this essay, QCD formalism is constructed almost completely from the underlying gauge symmetry. In formulating its lattice extension, this gauge symmetry has to be preserved. However, the gauge boson A_μ is defined at the limit $\epsilon \longrightarrow 0$. So it does not make any sense in the present context of lattice model. In fact, the more fundamental quantity is the comparator $U(y, x)$. We will define the comparators of nearest neighbors as

$$U(x, x + \hat{\mu}) = U_\mu(x) \tag{21}$$

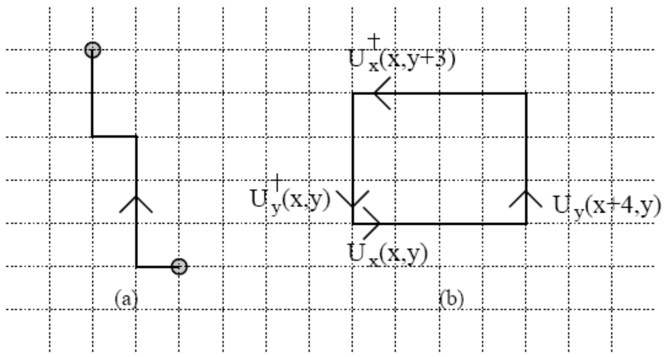

FIG. 6: Two gauge invariant quantities

There are only two types of gauge invariant terms that can appear in the Lagrangian

- A string linking a fermion and an anti-fermion consisting of path-ordered product of comparators, as show in Fig. 6(a)

$$\mathrm{tr}\bar{\psi}(x)U_\mu(x)U_\nu(x+\hat{\mu})\cdots U_\rho(y-\hat{\rho})\psi(y) \tag{22}$$

- A closed Wilson loop, as shown in Fig. 6(b). For example,

$$\mathrm{Re}\ \mathrm{tr}U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x) \tag{23}$$

The simplest form for gauge action is the above 1×1 Wilson loop $W_{\mu\nu}^{1\times 1}$ as Eq. (23). To make it identical with QCD in continuum limit, the action should take the form

$$S_g = \frac{2}{g^2}\sum_x\sum_{\mu<\nu}\mathrm{Re}\ \mathrm{tr}\ (1-W_{\mu\nu}^{1\times 1}) \tag{24}$$

A simple action involving fermions is

$$\bar{\psi}(x)\sum_\mu \gamma^\mu\left(U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\right) \tag{25}$$

Real computations are more complicated. We need to define the measure of path integral. When calculating correlations functions, we also need to a transcription of operators.

B. Results

In this subsection, we are going to see the lattice QCD results of deconfinement phase transition.

Fig. 7 shows the energy density ϵ rises steeply around $T = T_c$, confirming the existence of the proposed phase transition. Then it quickly reaches a plateau, about 70% of Stefan-Boltzmann limit, indicating the quark-gluon plasma are weakly coupled.

Fig. 8 shows chiral condensate $\langle\bar{\psi}\psi\rangle$ and $L(f) \propto \exp(-f/T)$ where f is the free energy, as functions of temperature T. When $T < T_c$, $\langle\bar{\psi}\psi\rangle$ is large manifesting chiral symmetry

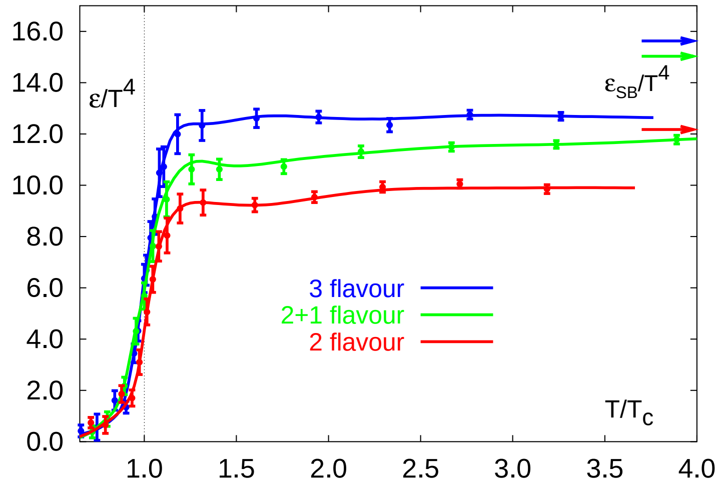

FIG. 7: Lattice simulation of energy density with temperature. The arrow indicates the position of Stefan-Boltzmann limit

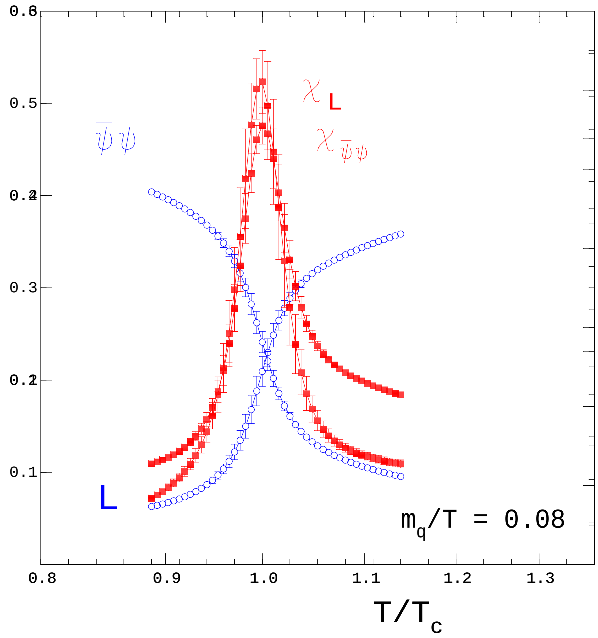

FIG. 8: chiral condensate $\langle\bar{\psi}\psi\rangle$ and free energy function $L(f)$ with temperature (blue). Their susceptibilities are shown in red.

breaking while L is small, telling us quarks are bound. At $T > T_c$, we do see deconfinement and restoration of chiral symmetry. We also plot their associated susceptibilities defined as $\chi_L = \langle L^2\rangle - \langle L\rangle^2$. They indicate the fluctuation of thermodynamic quantities. We do see a peak at $T = T_c$, which may diverge as system size tends to infinity. It is a strong evidence of second order phase transition.

IV. FINAL REMARKS

QCD, although based on simple gauge invariance principle, does possess a great variety of different phases. It is a place where different fields of physics meet together. We have already seen the BCS theory gives us the prediction of color superconductivity. In fact, QCD may have even stronger roles in condensed matter physics. Take the example of cuprates high-T_c superconductivity, which remain unsolved after more than 20 years of experimental discovery. In general, Hubbard model is believed to be the minimal model to describe the electrons on CuO_2 planes, where most of physics happens. Both QCD and Hubbard model are theories where traditional perturbation techniques break down. It is more amazing to note that they have a similar phase diagram. At $T = 0$ and half filling, Hubbard model describes a Mott insulator, where a charge gap forms, which is an analogy of chiral symmetry breaking and color confinement. Charge and color superconductivity do appear around the same place. People have used $SU(2)$ gauge symmetry to describe the spin liquid phase of doped Mott insulators. Developments in QCD may help us solve the mystery of high-T_c, and vice versa.

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[1] Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory.*

[2] Hildegard Meyer-Ortmanns, *Phase Transitions in Quantum Chromodynamics*, Rev. Mod. Phys. **68**, 473.

[3] Frank Wilczek, *QCD in Extreme Conditions*, hep-ph/0003183.

[4] Peter Arnold, *Quark-Gluon Plasmas and Thermalization*, arXiv:0708.0812.

[5] Simon Hands, *The Phase Diagram of QCD*, physics/0105022

[6] Rajan Gupta, *Introduction to Lattice QCD*, hep-lat/9807028.

[7] John B. Kogut, *An Introduction to Lattice Gauge Theory and Spin Systems*, Rev. Mod. Phys. **51**, 659

[8] Carleton DeTar, *Quark Gluon Plasma from Numerical Simulations of Lattice QCD*, hep-ph/9504325

[9] Frithjof Karsch, *Lattice QCD at Finite Temperature and Density*, hep-lat/9909006

[10] Helmut Satz, *The Thermodynamics of Quarks and Gluons*, arXiv:0803.1611

[11] David d'Enterria, *Quark-Gluon Matter*, nucl-ex/0611012

[12] Patrick A. Lee, Naoto Nagaosa, Xiao-Gang Wen, *Doping a Mott insulator: Physics of High-Temperature Superconductivity*, Rev. Mod. Phys. **78**, 17