Aspects of Universality in Quantum Hall Effect

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Abstract

In this essay, we intend to review aspects of universality and phase transition in the quantum Hall effect. In the perspective of scaling, we first notice the nature of the transition between two quantum Hall liquids, and briefly discuss theoretical understanding on it. In the fractional quantum Hall state, we review the theory of Laughlin's wavefunction, followed by the comment on universality class. In order to formulate this notion, an effective low energy field theory - the form of Landau-Ginzburg theory will be developed in terms of Chern-Simons gauge. It will then be shown that the most of known phenomenological features can be obtained from the mean field theory solution and the low energy fluctuation.

1 Introduction

A two dimensional electron system develops a striking set of phenomena when it is under a strong transverse magnetic field, at sufficiently low temperatures. In 1980, von Klitzing *et al.* [3] discovered that the Hall conductance σ_{xy} of a such system was very precisely quantized in an integer units of e^2/h, which is named Integer Quantum Hall Effect (IQHE). Two years later, Tsui *et al.* [4] repeated the same measurement in GaAs-$Al_xGa_{1-x}As$ heterojunctions, cleaner sample at lower temperatures. Surprisingly, it was observed that σ_{xy} can be a rational fraction of e^2/h, and such fractions are of some hierarchy. This kind of novel phenomena are usually called Fractional Quantum Hall Effect (FQHE).

Even though the two effects have a distinct microscopic origin, both are quite similar, at least macroscopically. On close inspection of the experiential results (Figure (1), Figure (2), and Figure (3)), we can notice the common defining features of both effects: (i) The Hall conductivity exhibits characteristic plateaus in a quantized way. (ii) At the same time, such region accompanies a high suppression of longitudinal conductivity, or equivalently, vanishing of dissipational restivity[1]. (iii) The transition between the two adjacent plateaus, driven by magnetic field, is described by certain type of scaling laws as a function of temperature. (iv) The filling fractions ν in the Hall conductance are observed only in some qualified rations and have some sequential order. (v) Near the filling fraction $\nu = \frac{1}{2s}$ ($s \in \mathbb{N}$), σ_{xy} is approximately equal to the classical value while σ_{xx} exhibits a minimum, which indicates the existence of Fermi surface, hence the metallic state.

Previously, it has been vaguely stated that the underlying mechanism of both QHE's are distinctive. However, the common phenomenological features take us to find a close connection between them and to search a unified picture. Before we proceed further, it is instructive to distinguish the microscopic origins of each states and to find some feasible connections. In the following section, we review the crucial mechanisms and find the connections of them, in the perspective of universality.

[1] The resistivity tensor, or equivalently conductivity tensor in two dimension is essentially off diagonal and independent of sample geometry. Hence the resistance(conductance) is essentially equal to the resistivity(conductivity).

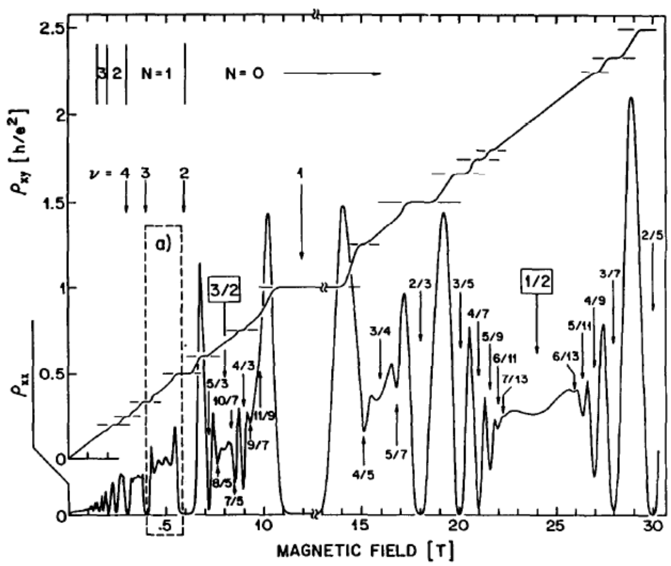

Figure 1: Transport data for the QHE: ρ_{xx} and ρ_{xy} as a function of B for a high mobility sample, which clearly exhibits both IQHE and FQHE. The data were taken at $T = 85mK$ with density $n_s = 3 \times 10^{11}/cm^2$ and a mobility $\mu = 1.3 \times 10^6 cm^2/V$. (from ref. [6])

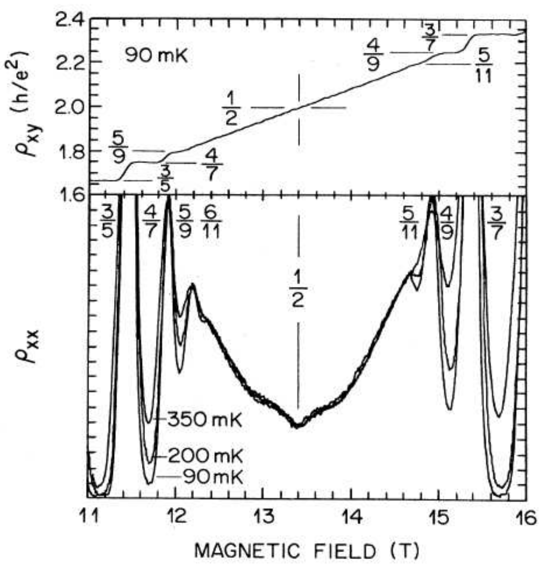

Figure 2: Temperature dependence of ρ_{xx} near filling factor $\nu = \frac{1}{2}$ (bottom) and the Hall resistance ρ_{xy} at T=90 mK (top) (from ref. [5])

2 Integer Quantum Hall Effect

2.1 Basic Mechanism and Gauge Argument

The basis understandings of the IQHE can be earned from the Landau level solution of the idealized system. In the noninteracting 2D electron system, one can easily check

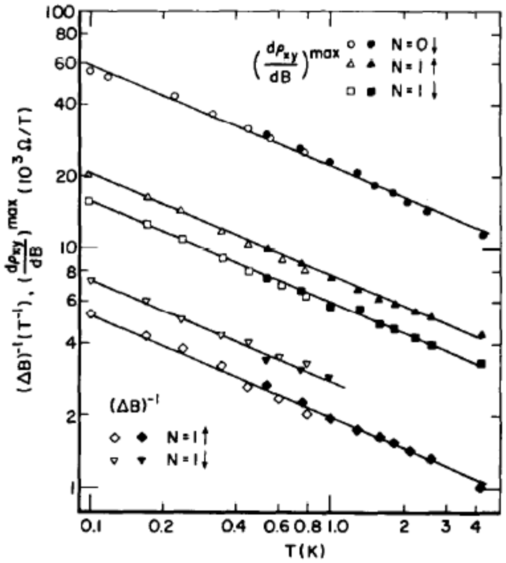

Figure 3: Temperature dependence of $(d\rho_{xy}/dB)^{\max}$ for the transitions: $\nu = 1 \rightarrow 2(N = 0 \downarrow)$, $\nu = 2 \rightarrow 3(N = 1 \uparrow)$, and $\nu = 3 \rightarrow 4(N = 1 \downarrow)$. The lower portion shows T dependence of $1/\Delta B$ for $N = 1 \uparrow, 1 \downarrow$. The slope of straight lines is 0.42 ± 0.04 (from ref. [6])

that the Hamiltonian is equivalent to a shifted harmonic oscillator. Since the degeneracy of each level are determined by the number of flux quanta, one can effectively move the Fermi level by tuning the magnetic field. When the Fermi surface lies between the gapped regions, the completly filled electrons has no room to dissipate so that it exhibits insulating behavior. As the Fermi level passes a Landau level, on the other hand, the increase in the transverse conductivity occurs in a quantized fashion. This picture provides accounts for the quantum Hall plateaus as well as non-dissipative characters in an elementary level.

In the real settings, however, a certain amount of disorder exists and it broadens to wipe the gapped dispersion. In consequence, most electrons are pinned to local impurities. Remarkably, electrons in the vicinity of the edges are essentially free of backscattering that erases the phase coherency, so that they remain extended. However, it may be still questionable how charge transport appears when the edges are spatially separated.

The key understanding on this was enhanced by the Laughlin's gauge argument. Since the quantization phenomena is insensitive to the continuous deformation of the sample, we can conceive ring shape geometry, whose inner part is threaded by magnetic flux. In principle, the Aharnov-Bohm phase acquired by the increase of the external flux can be gauged away. Even though the whole system is invariant, each individual state takes permutation that reproduces gauge invariance. The edge state, however, has no choice but to be pushed to the other Landau levels so it inevitably causes imbalance between edges. In order to resolve such thermal inequilibrium, the system makes a charge transported from one edge to the other, which is mediated by the delocalized states in the bulk lurked below the Fermi level.

For this reason, the IQHE is stable against the system specifics such as disorder and sample geometry (certain amount of perturbation). The natural appearance of gauge further suggests us that the QHE and the low energy properties are associated with the nonlocal features of gauge, say topological character. In the followings, we briefly see the scaling behaviors of plateaus transitions are indeed manifestation of renormalization in each topological sector.

2.2 Quantum Hall Phase Transition

The transitions between two quantum Hall liquids[2] are believed to be continuous. One strong evidence is from the experimental data, shown in Figure 3, which indicates the width of transition region continuously shrinks as a function of B, external control parameter. As the temperature goes to zero, it was also shown that transition width vanishes eventually.

These experimental observations lead us to assume that there exist a zero temperature critical point, where continuous phase transition occurs between the two quantum states. This implies that there is a correlation length ξ, which diverges as the external parameter approaches the critical value.

$$\xi \sim |B - B_c|^{\nu_\xi} \tag{2.1}$$

This correlation length corresponds to a localization length in a single particle description. In the interacting quantum system, there is a characteristic energy ω, which scales *via*

$$\omega \sim \xi^{-z} \sim |\Delta B|^{z\nu_\xi} \tag{2.2}$$

Following the standard scaling argument, we expect that the physically interesting quantities can be expressed in terms of scaling function, the arguments of which may be chosen to be the dimensionless ratios of the various energy scales and length scales.

$$\Gamma(\mathbf{k}, L, \omega, T, B) = \xi^{y_\Gamma} F(\mathbf{k}\xi, L/\xi, |\Delta B|^{z\nu_\xi}/T, \omega/T, \cdots) \tag{2.3}$$

[2]The name liquid comes from the incompressibility of the states, which can be directly checked from $\kappa \propto N^{-2} \left(\partial N/\partial \mu\right)_L$ at fixed angular momentum L. The presence of gap brings discontinuity in the chemical potential, hence incompressibility.

where y_Γ is the scaling dimension of Γ, L is the finite system size and F is a universal scaling function. In particular, the conductivities is known to to have $y_\sigma = 0$, so that it was conjecture to be universal in the limit $\xi \to 0$ (or, $B \to B_c$). In followings, we briefly quote the several approaches in order to justify (1) the existence of zero temperature critical point, (2) quantities of exponent ν_ξ, z and (3) σ_{ij}.

Numerical studies on this issue were mostly based on the noninteracting electrons in a transverse field, which is believed to be a close model to the plateaus transitions[9]. In the finite size system, the correlation amount to the size of the system near the transition point, hence we can associate the external parameter with the size L. Since the magnetic field bring the movement of the energy level, it is natural to replace ΔB with ΔE in the correlation scaling. Thus, $L \sim \xi(\Delta E)$, or $\Delta E \sim L^{-1/\nu_\xi}$. From the scaling behavior of the number of state $N \sim \Delta E/\delta$, where $\delta \sim L^{-2}$ is the two dimensional level spacing, we can see $N \sim L^{-1/\nu+2}$. All the numerical efforts on the finite size system, considering some random disorders, have found $\nu = 2.35 \pm 0.08$ at isolated energies. This supports the assumption of the existence of critical point. In addition to this, the longitudinal conductance turned out to be universal at transition $\sigma_{xx}^{\rm critical} = 0.5 \pm 0.02 e^2/h$.

These numerical results can be further augmented by the other theoretical works. A field theoretical approaches to the IQHE and the scaling behaviors were developed by Pruisken[10] and by Khmelnitskii[7]. The key features in Pruisken's approaches are (1) the correlation functions of noninteracting single particle can be effectively extracted from nonlinear sigma type model in $d = 2$ *via* replica method, (2) the presence of a strong magnetic field was incorporated into the action on the basis of symmetry argument by fixing the coupling constants, conductance. Quoting the Pruisken's action,

$$S[\mathbf{Q}] = \frac{1}{8}\int d^2x \left[\sigma_{11}{\rm tr}\big(\partial_\mu \mathbf{Q}\partial^\mu \mathbf{Q}\big) - \sigma_{12}\epsilon_{\mu\nu}{\rm tr}\big(\mathbf{Q}\partial_\mu \mathbf{Q}\partial_\nu \mathbf{Q}\big)\right] \tag{2.4}$$

where $Q_{ab}^{ss'} \sim \bar{\psi}_a^s(x)\psi_b^{s'}(x)$, the two fermion fields product. Notice the second term in the action corresponds to generalized winding number, topological in its origin. Extending this formulation, Khmelnitskii studied the functional expectations for σ_{11} and σ_{12} from the partition function, expanded in terms of the winding number index. To keep the consistency in the symmetry upon sample rotation ($\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = -\sigma_{21}$), the series expansion should be written as

$$\begin{aligned} \sigma_{11} &= \sigma_{11}^0 + \delta\sigma_{11} + \sum_{W=1}^{\infty} f_{W,\sigma_{11}^0}\cos(2\pi W\sigma_{12}^0) \\ \sigma_{12} &= \sigma_{12}^0 + \delta\sigma_{11} + \sum_{W=1}^{\infty} g_{W,\sigma_{11}^0}\sin(2\pi W\sigma_{12}^0) \end{aligned} \tag{2.5}$$

Note this emphasizes σ_{11} can be renormalized when $W = 0$, while the Hall conductivity σ_{12} remains unchanged. By Pruisken *et al.* the beta functions were derived for $W =$

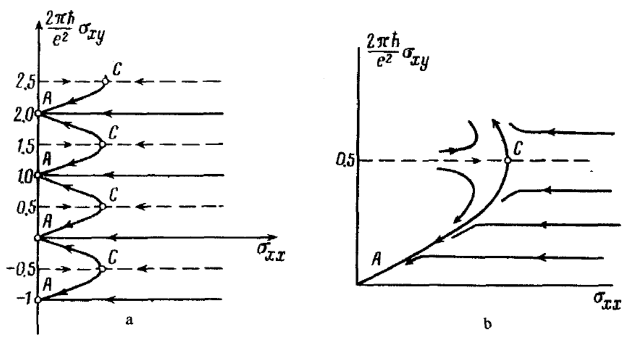

Figure 4: The renormalization flow diagrams for two parameters σ_{11} and σ_{12}. There are sinks $(\sigma_{11}, \sigma_{12}) = (0, \mathbb{N})$ for each sector and corresponding critical points $\sigma_{12} \in \mathbb{N} + \frac{1}{2}$. (from ref. [7])

$0, 1.$[10]

$$\beta_{11} = \frac{\partial \sigma_{11}}{\partial \ln L} = -\frac{1}{2\pi^2 \sigma_{11}} - c\sigma_{11} e^{-2\pi\sigma_{11}} \cos(2\pi\sigma_{12})$$
$$\beta_{12} = c\sigma_{11} e^{-2\pi\sigma_{11}} \sin(2\pi\sigma_{12}) \tag{2.6}$$

where $c > 0$ is numerical constant. The renormalization flow of the two coupled parameters are well illustrated in Figure 4. From eq (2.5) it follows there are two fixed points, one is sink and the other is critical fixed point. As to the first point, system flows into an quantum Hall liquid phase. On the critical surface at $\sigma_{12} \in \mathbb{N} + 1/2$, the longitudinal conductance scales towards a constant while σ_{12} does not renormalize so that metallic behavior appears.

This approach is very instructive in that it provides a microscopic account for the transitions between two integer quantum Hall liquids. By integrating out the short distance modes for each winding number modes, the standard renormalization argument seems to reproduce the features of transition between two integer quantum Hall liquids. However, it is questionable whether this is valid approach to the other states. In the following section, we will see field theoretical approaches to the FQHE, which is another way to see universal features in QHE.

3 Fractional Quantum Hall Effect

3.1 Laughlin Wavefunction

In the last section, we have noticed that the IQHE can be understood from the noninteracting electron picture, conspired with impurities. This frame is not applicable to the FQHE since Landau level is gapless to the fractional filling. Natural account necessarily requires interactions, which might be sensitive to the particular shape of electron interaction. By writing down the variational guess, Laughlin[11] was able to get remarkable approximation.

$$\psi_{2m+1}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^{2m+1} e^{-\sum_k=1^N |z_k|^2/4l^2} = \prod_{i<j}(z_i - z_j)^{2m}\psi_1(\{z_i\}) \tag{3.1}$$

where $z_j = x_j + iy_j$ is the complex coordinate of jth electron, l is the magnetic length, and ψ_1 is the lowest level Landau level wavefunction.

This form has significant feature in that the wavefunction is highly suppressed as any two particles approach each other. In case of symmetric gauge choice, the rotational symmetry requires the wavefunction has definite angular momentum, which is also manifest in the wavefunction. Counting the total angular momentum and using the formula of filling factor, we can easily check the above trial function brings a correct filling factor for primary sequences $\nu = \frac{1}{2m+1}$.[3] Above all, this form turns out to be very accurate ground state for some repulsive interaction as well as other types. Finally, the last equality in eq (3.1) can be seen as an attachment of Aharonov-Bohm phases due to the magnetic flux $2m\phi_0$ sitting on the particle. This observation plays an important role for the effective field theory of FQHE later.

To understand the transport property, we can repeat the same gauge invariance argument for the given wavefunction. If we imagine adiabatic insertion of unit flux in an infinitesimal hole at the center, we expect charge $\frac{e}{2m+1}$ to be transported from the center to the outer perimeter of the sample. Since the insertion of a flux returns the Hamiltonian *via* gauge transformation, the system move to an excited eigenstate, called 'quasihole' state. According to Laughlin, the quasihole located at z_0 can be written as

$$\psi^{\rm qh}_{2m+1}(\{z_j\}) = \prod_i (z_i - z_0)\psi_{2m+1}(\{z_j\}) \tag{3.2}$$

In this state, the angular momentum of each electron has been increased by one so it looks like the ground state of $N+1$ electrons, but with N electrons, one deficit at $z = z_0$. Hence,

[3] Total angular momentum $M = (2m+1)N(N-1)/2$, net area $A = 2\pi l^2 k_{\rm max} = 4\pi l^2 M/N$. The filling factor $\nu = 2\pi l^2 N/A = 1/(2m+1)$

the quasihole at z_0 has a fractional charge $\frac{e}{2m+1}$, existence of which was verified through shot noise measurement. Similar analysis can be used to construct a quasiparticle, but it is rather hard to analysis due to the difficulties of writing the wavefunction. The gap in the FQHE is thus naturally understood in terms of a separated quasiparticle-quasihole pair. The energy gap is then $\Delta = (\mu^+ - \mu^-)/(2m+1)$, where $\mu^- = E_0(N) - E_0(N-1)$ and $\mu^+ = E_0(N+1) - E_0(N)$, which is determined from the transport measurement, say $\rho_{xx} \sim e^{\Delta/k_BT}$. If the separation is not that wide, then the Coulomb-like attraction between them further reduces the gap and it also induces a circular motion, analogous to a roton in the superfluidity.

Thus far, the Laughlin wavefunction only explains the primary plateaus at $\nu = 1/(2m+1)$. To an extension, the other state can be thought of as descendants from the primary states. The idea of hierarchy was formulated by Haldane. The key idea is that the quasiparticles can condense themselves into a Laughlin-like state as the original electron did. Iterating this process, then, a set of filling factors can be connected. It should be here stressed that this approach is flawed in that it is hard to think of wavefunction and it is only meaningful for low energy quasiparticles.

Another perspective on this issue was obtained from the innocent looking equality in eq (3.1), which is named 'composite fermion' picture, suggested by Jain. Relaxing the condition in eq (3.1), one can attach the same factor to the other Landau level wavefunctions ψ_k and then project it into the lowest Landau level. Note this is equivalent to glow the $2m$ magnetic flux quanta to each electron so that it feels reduced magnetic field. Depending on the Landau level k, a set of filling factors $\nu = \frac{k}{2mk\pm1}$ is attained so that we may treat these in an equal footing.

The fact that Jain's approach maps the FQHE to the IQHE is quite remarkable. In other words, this is a mapping of an interacting fermion system to a noninteracting fermion system, while Haldane's view is to take the fermionic system to a bosonic system with flux attached that can form a condensate. The fact that the same phenomena can be viewed from either fermionic or bosonic viewpoint tells us that there must be a universal theory, forming an equivalent class.

3.2 The Chern-Simons-Landau-Ginzburg Theory for the QHE

In the previous section, we have discussed the features of FQHE in terms of Laughlin's wavefunction. However, it should be noted that wavefunction is not necessarily ground state for an arbitrary interaction. Then, we may ask why they works well. One reasonable answer is to regard them as representative of a universal class, which correspond to the fractional quantum Hall liquid state. In this perspective, we need to reexamine the wavefunction to see which fundamental properties are expressed through the wavefunctions. As repeatedly smelled throughout, the flux attachment is the most crucial feature, which immediately reformulated by a gauge. In order to find the aspect of universality, it is useful to write the effective low energy field theory for the QHE. One important point is the appearance of a gauge field, which is not local in its nature.

Not attempting to provide detailed derivations, we sketch the crucial points, following Zhang[12]. For a given interacting Hamiltonian, the Aharonov-Bohm phase for each electron can be removed by introducing a statistical gauge, $\mathbf{a}_j$. This process is equivalent to the unitary transformation of the Hamiltonian. Because of singular behavior, one can easily check this amounts to attaching flux quanta to each electron. Now, in terms of second quantization, it is straightforward to find an effective action. In the process, we can impose the supplementary constraint $\nabla \times \mathbf{a}(\mathbf{r}) = (2m+1)\phi_0 \bar{\psi}\psi = (2m+1)\phi_0 |\phi|^2$ by introducing the Lagrange multiplier field a_0.

$$Z = \int D\phi D\phi^* D\mathbf{a} Da_0 \; e^{-S} \; , \tag{3.3}$$

$$\begin{aligned} S = & \int_0^\beta d\tau \int d^2 r \phi^* (\partial_0 - ieA_0)\phi + \frac{1}{2m}\phi^* \big[-i\nabla - e(\mathbf{A}+\mathbf{a}) \big]\phi + \frac{1}{2}|\phi|^2 V |\phi|^2 \\ & + i \int_0^\beta d\tau \int d^2 r \; ea_0 (\nabla \times \mathbf{a}/(2m+1)\phi_0 - |\phi|^2) \end{aligned} \tag{3.4}$$

Though the introduction of a_0 makes the action invariant under gauge transformation, it is not that manifest. Temporarily relaxing the gauge choice of a^μ, we can eventually rewrite the second term in the action and then enforce the gauge fixing. Finally we have the chern-simons action.

$$S = S_0[\phi^*, \phi, A_\mu + a_\mu] + i \int d\tau dr \frac{e}{2(2m+1)\phi_0} \; \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{3.5}$$

Note that the statistical gauge field has no independent dynamics of its own, since there is no time derivative of a_μ in the action.

As in the Landau-Ginzburg theory of a superconductor, the bosonic field ϕ acquires long range order by breaking $U(1)$ symmetry of the gauge field $A_\mu + a_\mu$. In analogous to Meissner effect in the broken phase, it yields $B = -\nabla \times \mathbf{a} = 2\pi(2m+1)|\phi|^2$ *i.e.,* $\nu = 1/(2m+1)$. As a result of the Higgs mechanism, there is a gap to all excitations, as expected in FQHE. Finally, nontrivial topological excitation, say vortices are created as the magnetic field increase, which corresponds to a circular motion of quasiparticle-quasihole pair, as mentioned previously.

Note this approach is based on mapping the FQHE to a bosonic system, which carries both charge and flux quanta. Equivalently, the fermionic field theory can be developed from the composite fermion view. In the earlier part, we speculated these two picture should be same as long as they indicate the same phases of matter.

Another thing, we need to notice is the simultaneous existence of the charge and the flux, while superconductivity or superfluidity normally owns only one of them. This allows a dual field theory in which we change from particle to vortex variables, but preserves the form of the Chern-Simons action up to irrelevant terms with parameters redefined. It is important to realize that since dual transformation is exact and is invariant, we can repeat this procedure in which each dual field behaves as if fundamental particles. Hence, a family of filling factors are naturally described by a set of dual bosonic transformation, which accounts for a hierarchy.

4 Conclusion and Remarks

In this paper, we have sketched a few aspects of universality in quantum Hall effects. The phase transition between the two integer quantum Hall liquids is essentially zero temperature quantum phase transition. Parts of its features were captured in terms of standard scaling method, within each topological sector. However, a complete analytic theory of transition for general quantum Hall liquids is not established yet, though hierarchial structure seems to have some hints. In the FQHE, the surprisingly accurate wavefunctions take us to think of a universal class of state. This idea may be tested from the effective field theory. In bosonized theory, the fact that the fundamental field owns both charge and flux permits sequences of the dual transformation so that we could see the states fall into a class.

As alluded, the Chern-Simons action is related with topological invariants like Pruiskin's action. This is very interesting topic to study in terms of fractional statistics. Also the edge state, essentially $1+1$ dimensional object, allows a Luttinger liquid formulation. Like the $2+1$ dimensional case, the coexistence of charge and flux enables a dual transformation in $1+1$ dimension. The conformal characters in $1+1$ dimension further allows scaling consideration, which might be another interesting way of studying universality in QHE.

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