Dissipative Phase Transitions

Sarang Gopalakrishnan

May 5, 2008

Abstract

The transport properties of a quantum mechanical system coupled to an environment deviate sharply from those of the isolated system when the coupling exceeds a critical value. This effect is governed by a quantum phase transition, which takes place because the environment suppresses quantum fluctuations. This paper explains why dissipation leads to spontaneous symmetry breaking at $T = 0$, and how dissipative couplings stabilize superconductivity in Josephson junction arrays and nanowires.

1 Introduction

This paper is about the consequences of a simple fact—that a dissipative environment suppresses quantum tunneling. A naive explanation is that the system can't do anything quantum mechanical while the environment is looking at it. In a classic example of dissipative quantum mechanics, the "system" is a quantum particle in a double well potential, and the external world consists of (say) air molecules that intermittently bump into it. Since the air molecules fly off after the collision, and each air molecule possesses information about where it found the particle, quantum fluctuations must take place faster than τ, the timescale of interactions with the environment. (The air molecule picture is slightly misleading; the environmental coupling is not exclusively a thermal effect, and is important even at $T = 0$.) Whether the system is still able to tunnel depends on the ratio of τ to $t = \hbar/E$, where E is the barrier height. If $\tau \gg t$, the particle can sneak across the barrier without being observed; if $\tau \ll t$, it cannot.

Individual electrons and atoms can usually be treated as isolated; however, a mesoscopic or macroscopic quantum system presents a larger face to the world, and the suppression of quantum mechanics can have quite drastic effects. In superfluid (or superconducting) systems, the basic effect of the environment is to suppress quantum phase slips and thus stabilize superflow. (If it seems counterintuitive that dissipation makes a wire *more* superconducting, consider that anyone at rest on a grassy slope is being sustained in an unstable state by dissipative effects.) More generally, dissipation changes the critical behavior at quantum phase transitions, and sometimes introduces new phases.

So much for the physical picture: how are we to set up the problem of dissipation in quantum mechanics, let alone solve it? Section 2 introduces the Caldeira-Leggett model of quantum dissipation, and describes the simplest of the dissipative transitions—the localization of a quantum particle by the environment. Section 3 extends this analysis to finite-temperature behavior. Section 4 is a heuristic treatment of the $T = 0$ phase diagram of a Josephson junction array coupled to a bath of resistors. Section 5 reviews the extremely rich experimental developments in this field, and Section 6 suggests further applications of this paradigm in condensed matter and atomic physics.

2 Dissipative Localization at $T = 0$

2.1 The Caldeira-Leggett Model

In general, the world is described by the Hamiltonian $H = H_{system}(\psi) + H_{bath}(\chi) + H_{int}(\chi, \psi)$. At $T = 0$, the world is in its ground state: however, this is generally not a state of the form $|\psi\rangle_{sys}|\chi\rangle_{bath}$, because an interacting system is entangled with its bath. As the state of the bath is immaterial, what we are really interested in is the reduced density matrix of the system, $\rho_S = Tr_b \rho$, traced over the bath variables. For a realistic model of the bath, this trace is obviously intractable; however, one could plausibly expect that the details of the environment are irrelevant, and approximate it by a simple model. The Caldeira-Leggett model treats the environment as a bath of harmonic oscillators interacting linearly with the system, so the Lagrangian is:

$$L = L_S + \sum_\alpha \left[\frac{p_\alpha^2}{2m_\alpha} - \frac{1}{2} m_\alpha \omega_\alpha^2 x_\alpha^2 - \lambda_\alpha q - \frac{1}{2} \frac{q\lambda_\alpha}{m_\alpha \omega_\alpha^2} \right]. \tag{1}$$

(The last term compensates for the change in oscillator frequency due to the coupling.) This Lagrangian is quadratic in x_α; therefore x_α can be path integrated over, leaving an effective action of the form

$$S_{eff} = \int dt L_S(q(t)) + \int dt dt' F(q(t), q(t')).$$

The new term in the Hamiltonian is called an *influence functional*, and depends on the bath parameters. The next step is to coarse-grain the environment. As the bath parameters appear in F only through the density of oscillators

$$J(\omega) = \sum_\alpha \frac{\lambda_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha),$$

the influence functional can always be rewritten in terms of J. The general expression is:

$$F(q(t), q(t')) = \int d\omega J(\omega) \exp(-\omega|t - t'|) q(t) q(t').$$

Different choices of J lead to different physics. The most commonly used spectrum, to which we restrict ourselves, is $J(\omega) = \eta\omega$ (Ohmic dissipation); this form can be derived by equating the phenomenological classical expression for power dissipated ($P = \frac{1}{2}\eta \dot{q}^2$) with Fermi's golden rule.

2.2 The Dissipative Double Well

Caldeira and Leggett [1] first applied their model to show that dissipative effects suppress tunneling across a double well. In an isolated system, the tunneling probability is given by the WKB result $P \sim \exp(-\int \sqrt{V - E} dx/\hbar)$; in the presence of Ohmic dissipation, this becomes

$$P \sim \exp\left[-\frac{1}{\hbar} \left(A\sqrt{mV} a + B\eta a^2 \right) \right] \tag{2}$$

where A and B are dimensionless constants of order unity.

Any effect that suppresses tunneling between degenerate minima should favor spontaneous symmetry breaking. The question is whether the effect is actually strong enough to actually break the symmetry. The original Caldeira-Leggett argument suggests not because the particle will tunnel through the well given long enough. A more careful analysis [8] of the infinitesimally biased double well ($V = (x^2 - a^2)^2 + \epsilon x$) shows that one does *not* recover the symmetric ground state $|L\rangle + |R\rangle$ in the limit $\epsilon \to 0$, if the dissipation is strong. Instead, the particle is localized in either the left or the right well.

The approach of Ref. [8] is to map the dissipative double well onto the inverse-square Ising model, which undergoes a phase transition in one dimension [10]. The effective action for Ohmic dissipation is

$$S(\theta) = \int S_0 dt + \eta \int \int dt' \frac{1}{(t - t')^2} q(t) q(t').$$

What this result shows is that the bath generates an interaction between paths that is not only nonlocal but long-ranged in time. To proceed further, we should consider which paths contribute appreciably to the path integral. To zeroth order, these are the paths of stationary action—the static paths that sit at the bottom of the wells, and the "bounce" paths that spend most of their time at the bottom of one of the wells, but hop back and forth from time to time [4]. These paths have a higher energy than the static ones, but there are many more of them because the hops can take place at any time; this multiplicity of paths is the equivalent of entropy in the theory of quantum phase transitions. Since the quantum mechanical partition function is [3]

$$Z = \int Dq(\tau) \exp\left(- \int_0^{\beta\hbar} H(q, \dot{q}, \tau) d\tau \right),$$

the zero-temperature limit effectively acts as a thermodynamic limit in the τ direction, and our system is infinite in the τ dimension, which we shall sloppily refer to as time.

In the absence of dissipation the bounces are essentially noninteracting when they are far apart, and the ground state is delocalized for the same reason that the 1D Ising model is disordered—a bounce is energetically suppressed by a finite amount (the barrier height) but can take place whenever you like; therefore, given long enough the bounces (or spin flips) always win and the particle is just as likely to be in one minimum as in the other. However, this argument breaks down if the bounces have long-range interactions: when long-range interactions are sufficiently strong, the energy cost of bounces might be sufficient to block tunneling altogether. This mapping can be made precise [8], and implies that the ground state is indeed localized for $\eta > \eta_c(V, a)$, where the critical dissipation is a highly nontrivial function of the well depth and separation.

2.3 Quantum Brownian Motion

Let us now consider a periodic potential in one dimension, interacting with an Ohmic bath as above, so that $H_S = \frac{p^2}{2M} + U_0 \cos(2\pi q/q_0)$. In the absence of dissipation, the ground state is a delocalized Bloch state. However, as we shall see, the ground state abruptly becomes localized at a critical value of η. Apart from being a natural extension of our previous result, this phase transition has important physical consequences in superconducting wires.

It is helpful to do a little preliminary dimensional analysis. Suitable dimensionless variables are: $\phi(t) = 2\pi q(t)/q_0$, the dimensionless position; $\alpha = \eta q_0^2/2\pi\hbar$; and $V_0 = M q_0^2 U_0/(2\pi\hbar)^2$. Note that α and V_0 are precisely the dimensionless quantities that appear in Eq. 2. We also need a high-energy cutoff $\Lambda = 4\pi^2\hbar/M q_0^2$, the energy required to trap a particle in a single well. In terms of these variables, the effective energy (which the ground state minimizes) is given by

$$H = \frac{1}{2}\int \frac{d\omega}{2\pi}\left(\frac{\alpha}{2\pi}\mathbf{sign}(\omega) + \frac{\omega}{\Lambda}\right)\omega|\phi(\omega)|^2 - V_0\Lambda \int d\tau \cos\phi(\tau) \tag{3}$$

The term in ω/Λ is a soft high-energy cutoff, which penalizes high-frequency modes. It can be dropped if we restrict the ω integral to $|\omega| < \Lambda$. Ref. [2] performs a momentum-shell renormalization group analysis of this Hamiltonian, and finds the following recursive relation for V_0:

$$\frac{\partial V_0}{\partial l} = \frac{\alpha - 1}{\alpha} V_0(l) + o(V_0^3) \tag{4}$$

For $\alpha < 1$, V_0 renormalizes to zero and is an irrelevant perturbation. For $\alpha > 1$, V_0 grows exponentially. The RG equations for α are trivial: to second order, α does not change under the RG. Using a duality relation of Eq. 3, one can also compute the flows for large V_0; as we see from the flow diagram, V_0 flows to infinity for $\alpha > 1$ and to zero for $\alpha < 1$. $V_0 = \infty$ implies that tunneling is completely suppressed; this corresponds to localization.

2.4 Note on the Singularities

Where did the singularities come from? At $T = 0$, nonanalytic behavior potentially takes place when the ground state of a system changes non-analytically as a function of some parameter κ. Except in special cases, such non-analyticities are smoothed out in finite systems and one has avoided level crossings with a gap Δ, but $\Delta \to 0$ in the infinite system limit. The localization transition is in fact a quantum phase transition of the system-bath Hamiltonian, and depends on two limits—the continuous bath limit and the zero-temperature limit. In terms of the Caldeira-Leggett formalism, the continuous bath generates the long-range interaction, and the zero-temperature limit is a thermodynamic limit in time.

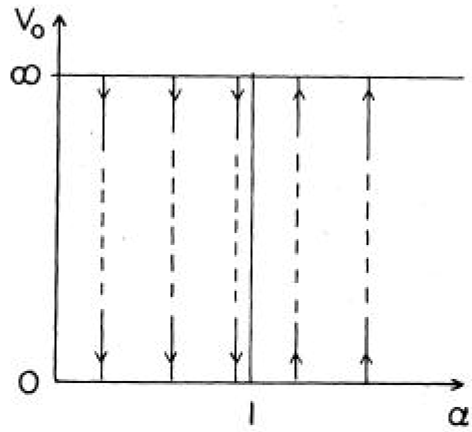

Figure 1: Renormalization Group Flows [2]

3 Finite-Temperature Effects

At finite temperatures, the quantum partition function of a single particle is an given by a path integral [3]:

$$Z = \int Dq(\tau) \exp\left(-\int_0^{\beta\hbar} H(q, \dot{q}, \tau) d\tau\right).$$

If we relabeled $q(\tau)$ as $\eta(x)$, this expression would look suspiciously like the coarse-grained one-dimensional Ising model in Landau theory. This suggests that the finite temperature case should be treatable by some form of finite size scaling—that the zero-temperature transition survives, in a rounded form, in the finite temperature system. The physical reason for the rounding out is simply that thermal hopping gives particles another way than tunneling to move about—and therefore, for example, the particle in the periodic potential eventually delocalizes no matter what η is. (This is an example of the fact that the order parameter is always zero in a finite system.)

One might nevertheless expect to see some sign of localization in the low-temperature correlation functions, and therefore (by the fluctuation-dissipation theorem) in the mobility or conductivity. Classically, a particle moving down a steady potential gradient Fx in the presence of friction reaches a steady velocity such that $v = \mu F$—friction exactly balances the potential gradient. Since $v = x(t)/t$ in a steady state, it follows that $\mu = x(t)/tF$ in this region. Fisher and Zwerger [2] use this fact to *define* the mobility as follows:

$$\mu = \lim_{t\to\infty} \frac{\langle x(t)\rangle}{tF}$$

The expectation value of $x(t)$ is simply $\langle x|\rho(t)|x\rangle$, and $\rho(t)$ can be computed by path integral techniques described in Ref. [3]. The eventual results for the mobility are plotted in Fig. 2.

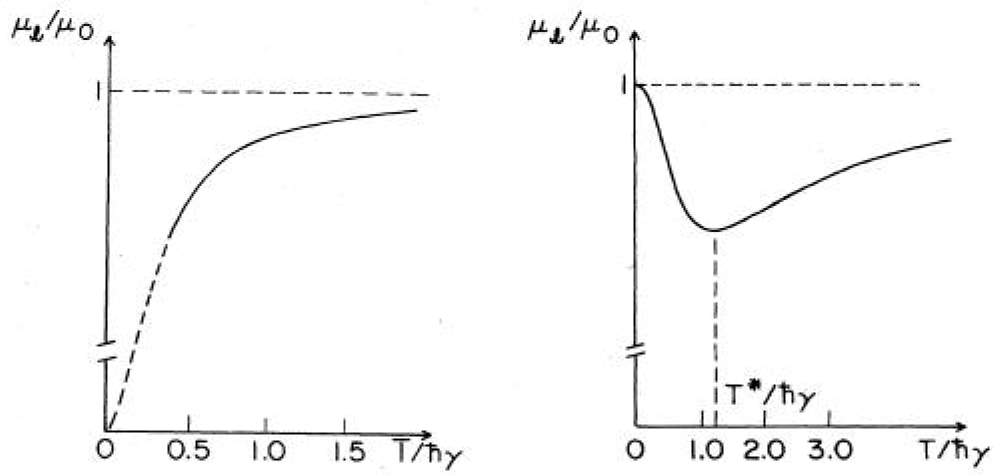

Figure 2: $\mu(T)$ for strong (L) and weak (R) dissipation

Initially localized wavefunctions become more mobile as you increase the temperature, because all their mobility is due to thermal hopping. On the other hand, initially delocalized wavefunctions become less mobile as you heat them up, because exchanging heat with the reservoir measures the system and destroys coherence, until a point T^* at which thermal hopping takes over. Note that, for any finite temperature, $\mu(T)$ is a smooth function of α: the only way to see a sharp transition is to go to $T = 0$.

4 Josephson Junctions

4.1 A Single Junction

A Josephson junction consists of two bulk superconductors (or superfluids) connected by a weak link [11]. Roughly speaking, the electron wavefunction in a superconductor has a well-defined phase ϕ, and both bulk wavefunctions seep into the weak link. This causes a phase difference across the link, which manifests itself in a current (since $j \propto \nabla\phi$). Suppose we crank up the phase difference across the junction in some continuous way (e.g. by applying a d.c. current): the phase gradient across the junction increases until at some point it winds by more than 2π. Since the bulk wavefunctions are only defined up to 2π, and phase twists cost energy, the junction can relax by shedding 2π of phase twist; however, it is in a metastable state because the phase can only relax if the wavefunction snaps and reforms in a less twisted state, a process known as a phase slip. A single Josephson junction with a large phase winding of $\Phi \gg 2\pi$ across it is a particle in a tilted washboard potential of the kind discussed above, because the energy has infinitely many local minima spaced a distance

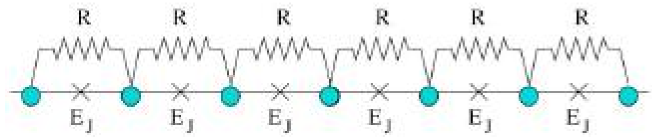

Figure 3: Resistively shunted Josephson junction array [5]

2π apart in Φ-space. (Note that Φ here is the total phase twist across the junction, and therefore is *not* a periodic variable, unlike $\phi(x)$, the local phase of the wavefunction.)

The results of Section 3, translated into supercurrent language, then imply that supercurrents are *more* stable for larger dissipative coupling; and that, for weak dissipation, supercurrents are *more* stable as you increase the temperature from $T = 0$ to some $T = T^*$—the environment suppresses quantum phase slips. As far as a single junction is concerned, this is the entire (and now fairly old) story.

4.2 Resistively Shunted Arrays

We now proceed to study one-dimensional arrays of superconducting grains connected by weak links.[1] In general, each weak link is permeable both to normal electrons (as a resistor) and to Cooper pairs (as a Josephson junction); therefore, the array can be represented in circuit form as in the figure. Apart from being a natural extension of our previous analysis, this system is interesting because, in the continuum limit, it describes a superconducting nanowire.

If there were no shunts ($R_S = \infty$), the phase structure would be determined by the interplay between the Coulomb repulsion between Cooper pairs E_C, which tends to localize them on individual grains, and the energy associated with phase coherence E_J, which tends to delocalize them across the sample. In this regime, all the conduction is due to Cooper pairs because the junction does not allow single electrons through. Adding a shunt resistance changes this because (1) it provides a channel for dissipation, (2) Cooper pairs can now break up and move through the wires (or, conversely, normal electrons can pair up and tunnel through the junction, depending on which process is easier). This adds two new variables to the phase diagram—R_S, the shunt resistance, and r, the "conversion" resistance, which is a phenomenological measure of how hard it is to convert normal electrons into Cooper pairs and vice versa. r is related to the size of the superconducting grains.

Dissipation must be stronger when more electrons flow through the resistor; therefore it must be inversely related to R_S and to r. The correct dimensionless value is $\alpha = R_Q/\sqrt{R_S^2 + 4rR_S}$. The other important dimensionless variable in this problem is the dimensionless phase stiffness $K = E_J/E_C$, which governs the superconductor-insulator transition

[1] A one-dimensional array is of course a chain, but the phrase "chains of grains" is impossibly ugly.

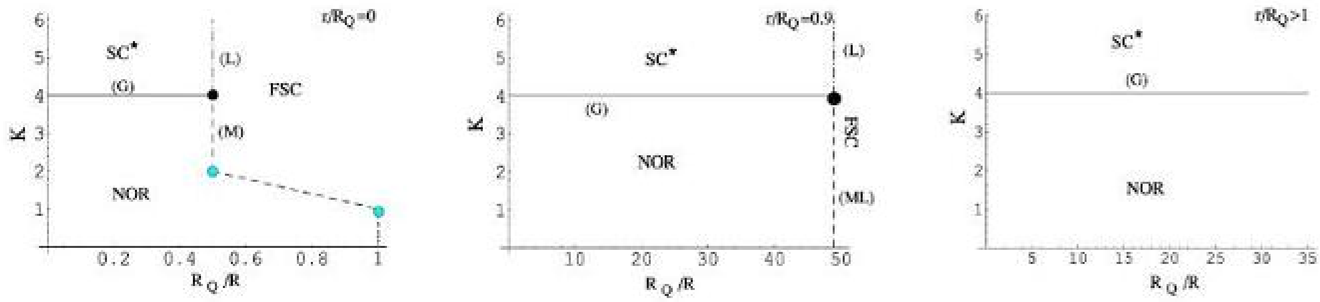

Figure 4: Phase Diagram of the JJ array for three values of grain size (largest L, smallest R). Note the disappearance of the FSC phase. Diagram reproduced from [5].

at zero dissipation.

The zero-temperature phase diagram of this system is quite rich. In the absence of dissipation, a zero-temperature quantum mechanical system in d dimensions behaves like a $d+1$-dimensional classical system, because of the additional integral over τ in the partition function. (See Section 3 above.) Therefore, a 1D quantum system with a complex order parameter behaves somewhat like the XY model in two dimensions. The classical XY model has no long range order but two phases, a low-temperature phase with power-law correlations and a high-temperature phase with exponentially decaying correlations. Long-range order is impossible because, for any $T > 0$, adding a vortex-antivortex pair decreases the free energy. When $k_B T \sim E_b$, the binding energy of a vortex-antivortex pair, the pairs dissociate; this process is called a Kosterlitz-Thouless transition.

For weak dissipation, this is the entire zero-temperature phase diagram—a superconducting phase with power-law correlations (labeled SC*), a metallic phase with exponentially decaying correlations (N), and a Kosterlitz-Thouless transition. The vortex pairs in the classical system map onto phase slip dipoles, which consist of a paired phase slip and anti-phase slip (i.e. a jump across the double well and back again), localized in spacetime. However, the dissipative coupling obstructs quantum phase slips and locks the phase differences between neighboring Josephson junctions. This leads to a new phase with phase differences pinned in time, the fully superconducting (FSC) phase. Like the SC* phase, it has algebraically decaying correlations; however, as the phases are locally locked in place, the order parameter locally has a nonzero expectation value. The phase diagram is plotted below for various values of the conversion resistance r; note that the FSC phase encroaches further into the normal phase than the SC* phase does, as a result of the dissipative stabilization we saw in Section 2.

There are now two more phase transitions in the diagram: the localization transition of a single Josephson junction, as you change the dissipation; and the transition from the dissipatively locked superconductor to a metal, as you change the superconducting stiffness.

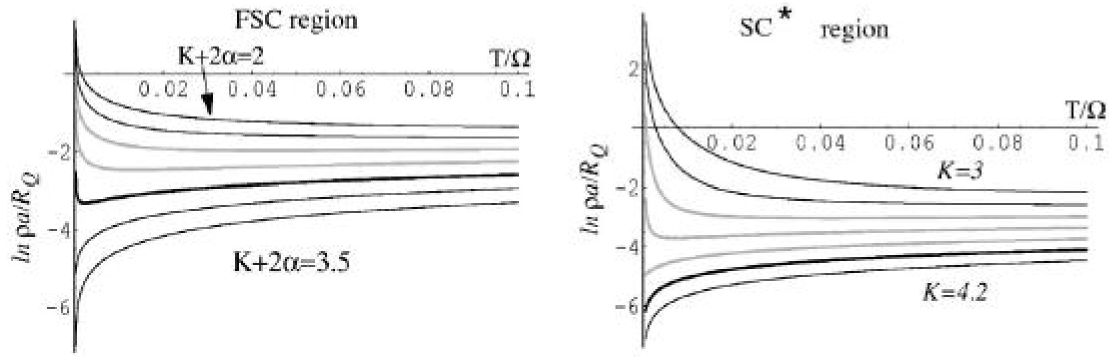

Figure 5: Resistivity vs. Temperature [5]

Unlike most phase transitions, these are driven by *local* effects that are emphatically not long-wavelength fluctuations. The SC*-FSC transition is entirely local in character, and is adequately described as being N identical junctions going through the washboard transition of Section 2. On the other hand, the transition from FSC to N has both local and global characteristics—it involves the depinning *and* dissociation of phase slip pairs. The behavior at this transition is still an open question, as there are two rather different limits that may or may not be separated by a multicritical point. The first is the transition labeled by ML in the diagram, as you tune the dissipation below criticality—it's a local transition, in which the phase variable at each junction delocalizes, followed immediately by a global Kosterlitz-Thouless transition, as the now unstable phase slip pairs dissociate. The second is the transition labeled by M, as you decrease the stiffness at constant shunt resistance for small conversion resistance r; for large r, this transition is forced down to $K = 0$. At the M transition, the individual junctions are still localized; however, the phase stiffness is too weak to maintain a correlation between the phases on neighboring junctions.

4.3 Finite Temperature Effects

As we saw in Section 3, dissipative transitions govern the low-temperature behavior of transport coefficients—in this case, the resistivity ρ. The plots below are theoretical predictions of $\rho(T)$ for various values of the phase stiffness. Superficially this graph shows the same nonmonotonic behavior that we saw previously. There are three important features—(1) There is a range in which the resistivity is almost flat or drops slightly for a wide range of temperatures. This behavior is generic in superfluid-insulator transitions, and has been observed in various experiments, prompting talk of an intermediate "metallic" phase at zero temperature [16]. According to the RG analysis of Ref. [5], there is no metallic phase at zero temperature in this system. (2) The critical curves are given by $K + 2\alpha = 3$ in the

FSC regime, and $K = 4$ in the SC* regime. So the crossover is entirely insensitive to α in a certain parameter range, and strongly sensitive in another. (3) The resistivity initially drops in a certain parameter range because thermal phase slips are suppressed; it initially rises for small K because the Cooper pairs localize on individual grains, and thermal hopping is suppressed. These results should all be straightforward to test experimentally; however, there appears to be little conclusive data on Josephson junction arrays.[2]

5 Nanowires

A nanowire is the continuum limit of a Josephson junction array, and one might naively expect the previous analysis to apply. It turns out that this limit is pathological for various reasons; however, there is strong experimental evidence [6, 7] for superconductor-insulator transitions in nanowires as one changes their diameter (which is proportional to phase stiffness: a phase slip is like a bubble, and relaxes when it reaches the edge of the wire, which is harder when the wire is thick). It has been argued [5] that the FSC phase cannot exist in nanowires, in which case the effects of dissipation would be fairly boring (see the phase diagram in Section 4).

The situation is somewhat better for finite (i.e. short) wires. A surprising feature of these wires is the anti-proximity effect [12]: when two bulk superconducting leads are connected by a short (2μm) zinc nanowire, the nanowire is resistive at all measured temperatures. However, if you destroy the superconductivity of the leads by applying a magnetic field,[3] the nanowire is superconducting below 0.8 K. Fu et al [13] explained this effect as follows—a bulk superconductor screens vortex-antivortex interactions and unbinds vortex pairs; a resistive lead, on the other hand, suppresses vortex mobility near the edge, and stabilizes supercurrents. In the path integral formalism of Section 2, a finite wire maps onto a strip in spacetime; the superconducting leads act as mirrors for phase slips, which interact with their images, and these interactions dissociate dipoles and allow phase slips to dissociate.

The case of resistive leads is harder. For very short wires, [7, 15] there is (somewhat mixed) experimental evidence for a sharp superconductor-insulator transition at a critical value of the *total resistance* of the wire; for longer wires, this appears not to be the case. Besides, the temperature dependence of the resistivity is much sharper than the predictions of the theory described in Section 4. The theory breaks down because interactions between phase slips and the leads change the shunt resistance (dissipation) that other phase slips feel. Meidan et al [14] account for this effect in the mean-field approximation, which appears adequate to explain the data.

[2]For a very thorough bibliography on Josephson junction arrays, see Ref. [5].

[3]The applied field must be greater than the critical field of the leads, but less than the critical field of the nanowire.

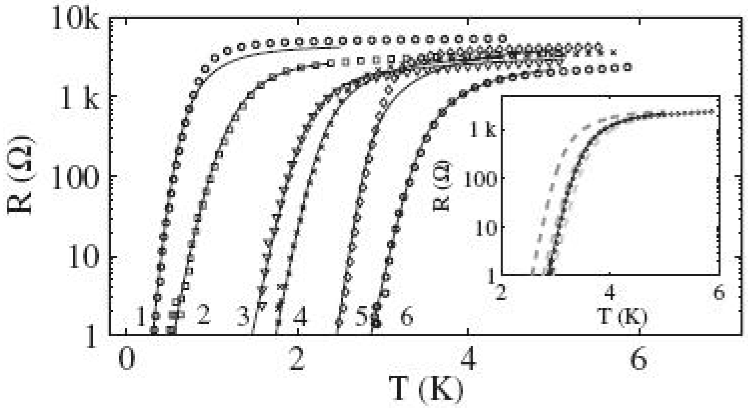

Figure 6: Resistance vs. Temperature in Nanowires [14]

6 Conclusions and Outlook

Dissipative transitions are quantum phase transitions that govern the low-temperature transport properties of various systems. We have focused on the stabilization of superfluidity in low-dimensional systems, but there are other areas—quantum Hall transitions and itinerant magnetism, for instance—in which dissipative transitions are thought to occur.[4] Two relatively direct extensions of our previous analysis are to disordered thin films, in which the superconducting transition takes place along percolating paths through the system and is therefore one-dimensional, and atomic Bose-Einstein condensates interacting with a cloud of excitations (or multiple-species atomic clouds). Cold atom systems are particularly promising because they are naturally isolated to a much greater extent than condensed matter systems, and one can therefore "turn on" dissipation much more controllably, and even potentially engineer the spectrum of the bath in order to stabilize certain states (e.g. if dissipation localized the site occupation number instead of the phase, it would stabilize a Mott insulator) [17].

It has also been suggested [17] that dissipative effects in *driven* systems can help to stabilize certain quantum states. The idea is that either the friction or the driving knocks the system out of every state except the so-called dark state; as a result, every state ultimately decays into the dark state. Apart from potential applications in quantum state engineering, driven quantum systems can be used to study nonequilibrium quantum phase transitions, and to investigate fundamental questions about quantum coherence and quantum information.

[4]See Kapitulnik et al [16] for a detailed list.

References

[1] A.O. Caldeira and A.J. Leggett, Phys. Rev. Lett. 46, 211 (1981)

[2] M.P.A. Fisher and W. Zwerger, Phys. Rev. B32, 6190 (1985)

[3] R.P. Feynman, *Statistical Mechanics*, Westview Press (1972)

[4] A. Altland and B. Simons, *Condensed Matter Field Theory*, Cambridge U. Press (2006)

[5] G. Refael, E. Demler, Y. Oreg, and D.S. Fisher, Phys. Rev. B75, 014522 (2007)

[6] A. Bezryadin, J. Phys.: Cond. Mat. 20, 043202 (2008)

[7] C.N. Lau, N. Markovic, M. Bockrath, A. Bezryadin, and M. Tinkham, Phys. Rev. Lett. 87, 217003 (2001)

[8] S. Chakravarty, Phys. Rev. Lett. 49, 681 (1982)

[9] P. Goswami and S. Chakravarty, Phys. Rev. B 73, 094516 (2006)

[10] J. Fröhlich and T. Spencer, Commun. Math. Phys. 84, 87 (1982)

[11] A.J. Leggett, *Quantum Liquids*, Oxford U. Press (2006)

[12] M. Tian, N. Kumar, S. Xu, J. Wang, J.S. Kurtz, and M.H.W. Chan, Phys. Rev. Lett. 95, 076802 (2005)

[13] H.C. Fu, A. Seidel, J. Clarke, and D.-H. Lee, Phys. Rev. Lett. 96, 157005 (2006)

[14] D. Meidan, Y. Oreg, and G. Refael, Phys. Rev. Lett. 98, 187001 (2007)

[15] G. Refael, E. Demler, and Y. Oreg, e-print arxiv:0803.2515v1 (2008)

[16] A. Kapitulnik, N. Mason, S.A. Kivelson, and S. Chakravarty, Phys. Rev. B 63, 125322 (2001)

[17] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H.P. Büchler, and P. Zoller, e-print arxiv:0803.1482v1 (2008)