Renormalization in Complex Networks

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Abstract:
Complex networks are ubiquitous both in biology and sociology, and also widely constructed and exploited in science and technology. Common topological features are shared among various complex networks across disciplines. Renormalization of complex networks, yielding scaling laws and critical exponents, offers a new approach to category distinctive networks into universality classes, revealing similarities hard to capture in usually ways, whose analysis will help the understanding and facilitate improvement in different fields.

Introduction

Network structure, either in a concrete or abstract sense, is ubiquitous across various disciplines. Complex networks are widely observed and constructed in nature and sociology for better description and revealment of the underlying organization principles, and also broadly designed and exploited in science and technology for efficiency and economy in construction, maintenance and further spread into other districts.

In nature, food chain and web is a good example of complex networks, which may incorporate millions of species in different domains of life across wide areas in the biosphere. Linked by the predator-prey relationship, a wide range of species weaves into a complex network with directions. Chemicals in a cell linked by chemical reactions, protein-protein interaction network, and metabolic system of *E. coli*[1] are other examples with great study interest nowadays.

In sociology, every individual is linked with his or her acquaintances, which expands an undirected network. It is interesting to note here an amazing empirical study[2] by psychologist Stanley Milgram that every two people in the United States has an average acquaintanceship of six, i.e. on average any two are indirectly known through five other people (Fig. 1), which is later termed as "six degrees of separation"[4]. Although the acquaintance length may not be accurate, its smallness is later verified[5] and generally accepted[6]. Similar results are also obtained through collaboration relationship for actors[7] whether two have acted in a movie together, and for scientists[8] whether two have written a paper together.

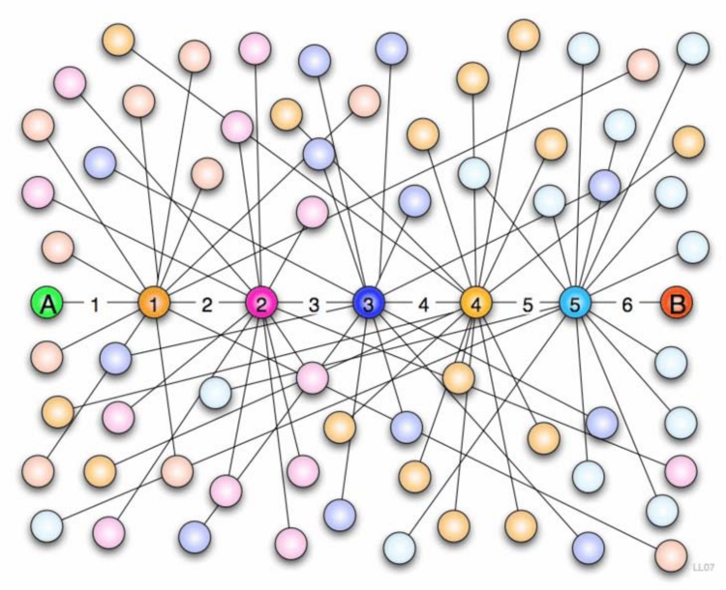

Fig. 1[3] Illustration of six degrees of separation: each small circle represents an individual. A pair is linked if they are acquaintances. "A" and "B" are indirectly connected through 1-5.

In the field of technology, the Internet is a physical network, where numerous routers and computers are linked through cables or wireless signals, and the World

Wide Web is a virtual network, where enormous web pages are directionally connected through hyperlinks.

We can sip the complexity of the networks from the examples listed above. The study of network structure falls into the field of graph theory, where various networks are simplified into nodes connected by edges. Mathematicians Paul Erdős and Alfréd Rényi proposed a random graph model[9], which serves as the mean field theory in physics. In the Erdős-Rényi model, each pair of nodes is randomly connected with some probability p. Thus in a random graph of N nodes, there are pN (N - 1) / 2 edges. It can be speculated that the real networks have underlying organization principles coded into their topological structures, which forecasts intrinsic deviations from the Erdős-Rényi model.

Not embedded in Euclidean space, complex networks are delineated in concepts from graph theory, among which there are three paramount intrinsic properties[10]: small-world, clustering and scale-free, which are described by average path length, clustering coefficient and degree distribution, respectively. First, the path length between any two nodes is the smallest number of edges connecting the pair. The "six degrees of separation" is the folklore version of the small-world feature, which emphasizes the smallness of the average path length. Erdős-Rényi model is small-world, too, where the average path length l increases only logarithmically with the number of the nodes N,

$l \approx$ lnN.

Second, the clustering coefficient of the whole graph is the average of the clustering coefficient of every node, which is the ratio of the real and maximal edges of the subgraph involving the selected node and its directly connected neighbors. The clustering coefficient of real networks is usually much larger, i.e. clustering, than that of Erdős-Rényi model (Table 1).

Network	Size	$\langle k \rangle$	ℓ	ℓ_{rand}	C	C_{rand}	Reference
WWW, site level, undir.	153 127	35.21	3.1	3.35	0.1078	0.00023	Adamic, 1999
Internet, domain level	3015–6209	3.52–4.11	3.7–3.76	6.36–6.18	0.18–0.3	0.001	Yook *et al.*, 2001a, Pastor-Satorras *et al.*, 2001
Movie actors	225 226	61	3.65	2.99	0.79	0.00027	Watts and Strogatz, 1998
LANL co-authorship	52 909	9.7	5.9	4.79	0.43	1.8×10^{-4}	Newman, 2001a, 2001b, 2001c
MEDLINE co-authorship	1 520 251	18.1	4.6	4.91	0.066	1.1×10^{-5}	Newman, 2001a, 2001b, 2001c
SPIRES co-authorship	56 627	173	4.0	2.12	0.726	0.003	Newman, 2001a, 2001b, 2001c
NCSTRL co-authorship	11 994	3.59	9.7	7.34	0.496	3×10^{-4}	Newman, 2001a, 2001b, 2001c
Math. co-authorship	70 975	3.9	9.5	8.2	0.59	5.4×10^{-5}	Barabási *et al.*, 2001
Neurosci. co-authorship	209 293	11.5	6	5.01	0.76	5.5×10^{-5}	Barabási *et al.*, 2001
E. coli, substrate graph	282	7.35	2.9	3.04	0.32	0.026	Wagner and Fell, 2000
E. coli, reaction graph	315	28.3	2.62	1.98	0.59	0.09	Wagner and Fell, 2000
Ythan estuary food web	134	8.7	2.43	2.26	0.22	0.06	Montoya and Solé, 2000
Silwood Park food web	154	4.75	3.40	3.23	0.15	0.03	Montoya and Solé, 2000
Words, co-occurrence	460.902	70.13	2.67	3.03	0.437	0.0001	Ferrer i Cancho and Solé, 2001
Words, synonyms	22 311	13.48	4.5	3.84	0.7	0.0006	Yook *et al.*, 2001b
Power grid	4941	2.67	18.7	12.4	0.08	0.005	Watts and Strogatz, 1998
C. Elegans	282	14	2.65	2.25	0.28	0.05	Watts and Strogatz, 1998

Table 1[10] Comparison between real networks and random graphs:
Size denotes the number of nodes, $<k>$ for the average degree, l for the average path length, and C for clustering coefficient.

Finally, the degree distribution $P(k)$ is the probability for a randomly chosen node with exactly k edges. While Erdős-Rényi model displays a Poisson distribution, real networks usually have a power-law tail, which is referred to as scale-free,

$$P(k) \approx k^{-\gamma}.$$

Although clustering coefficient and degree distribution manifest the non-random character for real networks, we still can not see clearly the difference in organization principles for distinctive real networks. Recently, C. Song, S. Havlin and H. A. Hakse[11] reported self-similarity of some complex networks under renormalization procedures. When the network is tiled by box-counting method with different box size l_B, the degree distribution for World Wide Web is invariant (Fig. 2). Scaling laws and critical exponents are thus discovered[12] in sequence and may be used to category various networks into universality classes, where common organization principles underlie.

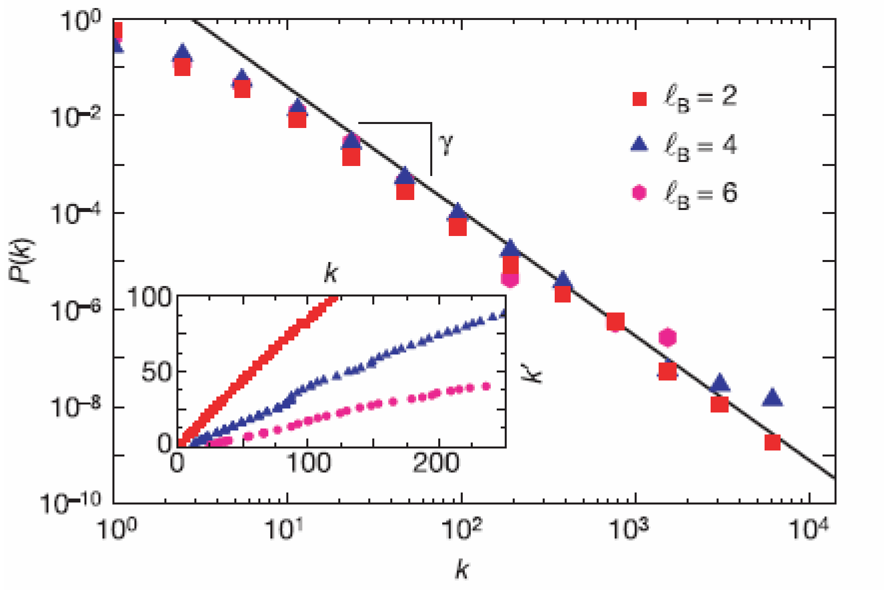

Fig. 2[11] Invariance of degree distribution for World Wide Web with different box size l_B. The inset shows the scaling of degree k.

Renormalization of Complex Networks

(1) Renormalization Procedure

Let us now consider a complex network G_0 with N_0 nodes connected by E_0 edges under a renormalization transformation R_{l_B} which coarse-grains the network. The coarse-graining procedure is performed in real space by the box-covering method[11](Fig. 3). We will tile the whole graph with boxes of box size l_B, i.e. the length between every pair of nodes inside the box is smaller than l_B. Two boxes are linked if there is at least one edge connecting the nodes in the two boxes. Then every box is replaced with a new node, which yields a new graph. The edges in the renormalized graph are the links between the previous boxes. After t successive renormalization transformations, we will have a renormalized graph

$$G_t = R_{l_B}^{\ t}(G_0),$$

with N_t nodes and E_t edges.

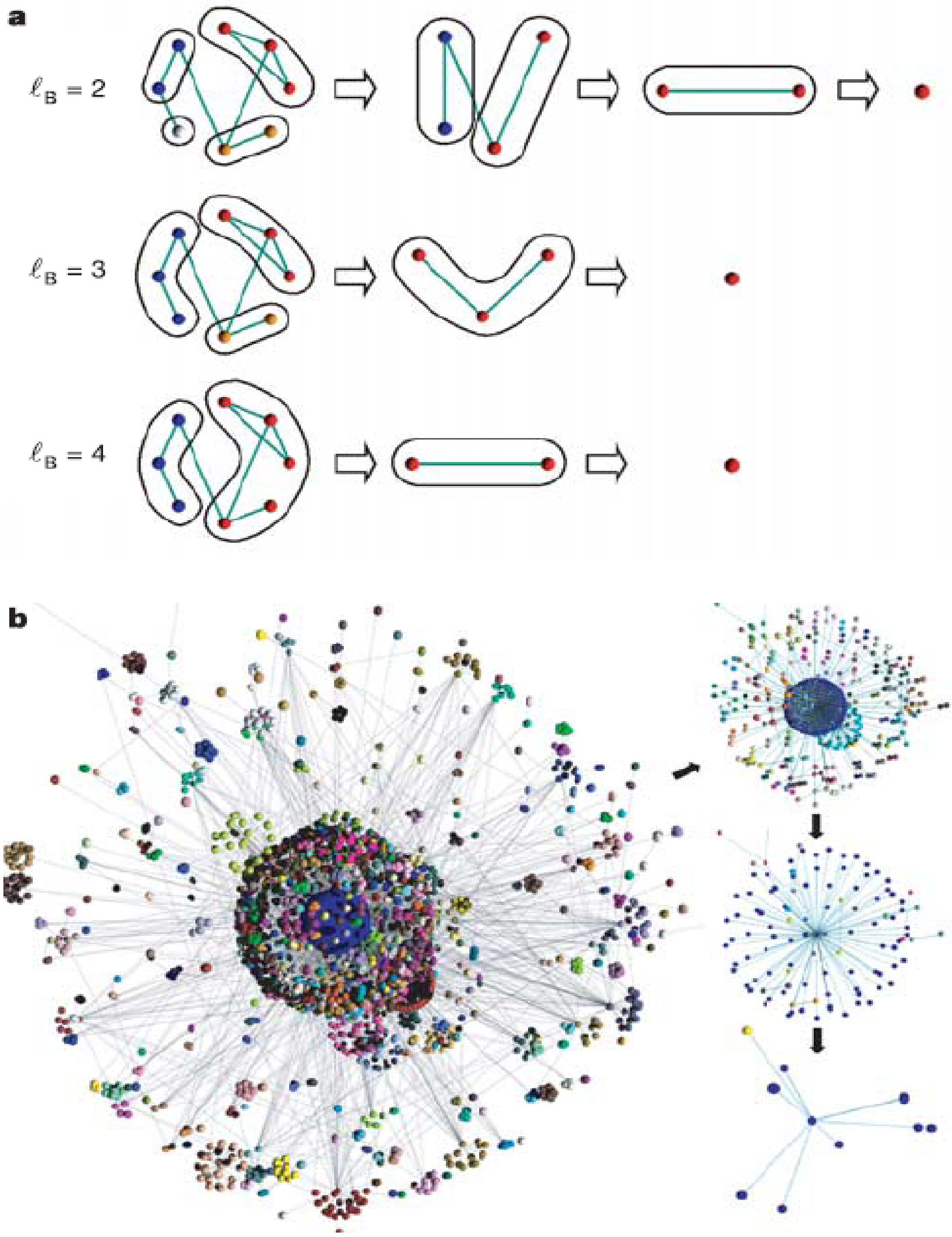

Fig. 3[11] Renormalization procedure by box-counting method.
a. Tile the network with different box size l_B.
b. Tile World Wide Web with l_B=3.

Since degree distribution characterizes networks, the flow of the relative maximum degree[12]

$\kappa_t = K_t / (N_t - 1)$,

where K_t is the maximum degree in graph G_t, and the average degree $2\eta_t$, where

$\eta_t = E_t / (N_t - 1)$,

will be tracked with the relative size of the renormalized network

$x_t = N_t / N_0$.

(2) Scaling Laws and Critical Exponents

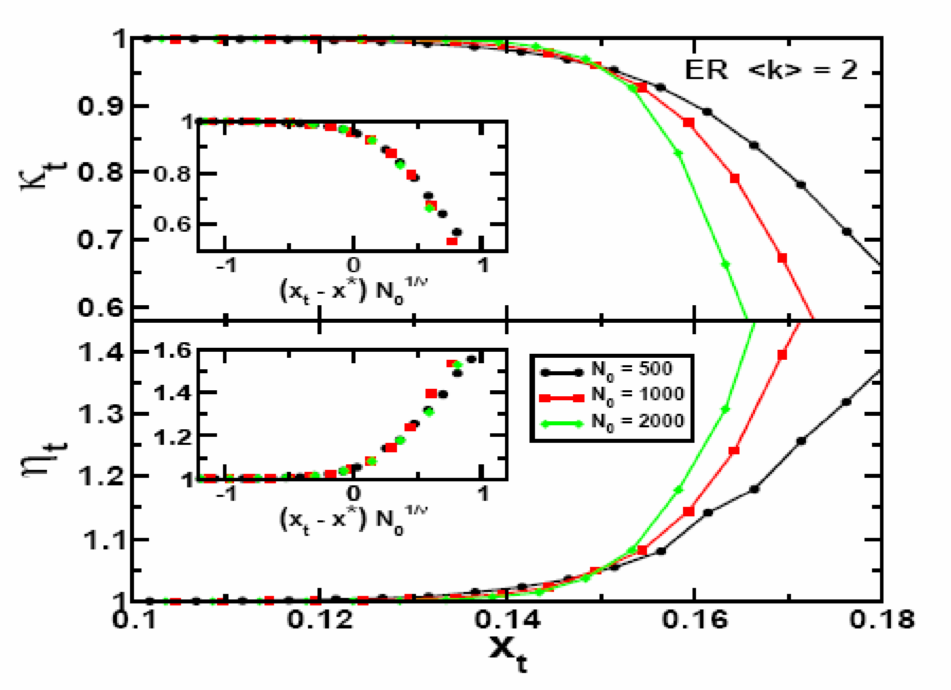

Fig. 4[12] Scaling behavior for the Erdős-Rényi model.

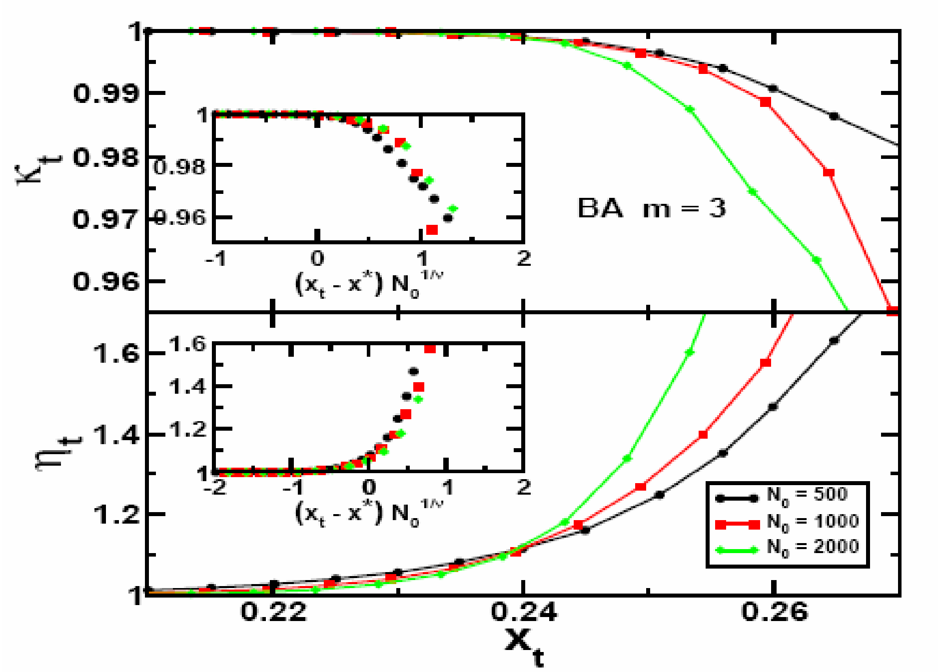

Fig. 5[12] Scaling behavior for the Barabási-Albert model.

Fig. 4[12] shows the flow of the relative maximum degree κ_t and average degree $2\eta_t$ with the relative size of the renormalized network x_t for different initial graph sizes N_0 in the Erdős-Rényi model with an average degree of 2. The point all the curves pass, which is the finite size effect, corresponds to the fixed point for initial graph with infinite size. Plotted as a function of $(x_t - x^*) N_0^{1/\upsilon}$ in the inset, the data have a remarkable collapse with the exponent $\upsilon = 2$. It is interesting to note that the critical exponent for κ_t and η_t are the same. Similar results are shown in Fig. 5[12] for the Barabási-Albert model[13], which is not self-similar, either, with the same scaling laws and critical exponents as the Erdős-Rényi model. It is suggested in [12] that generally non-self-similar graphs have scaling laws $\kappa_t \sim (x_t - x^*) N_0^{1/\upsilon}$ and $\eta_t \sim (x_t - x^*) N_0^{1/\upsilon}$ with $\upsilon = 2$.

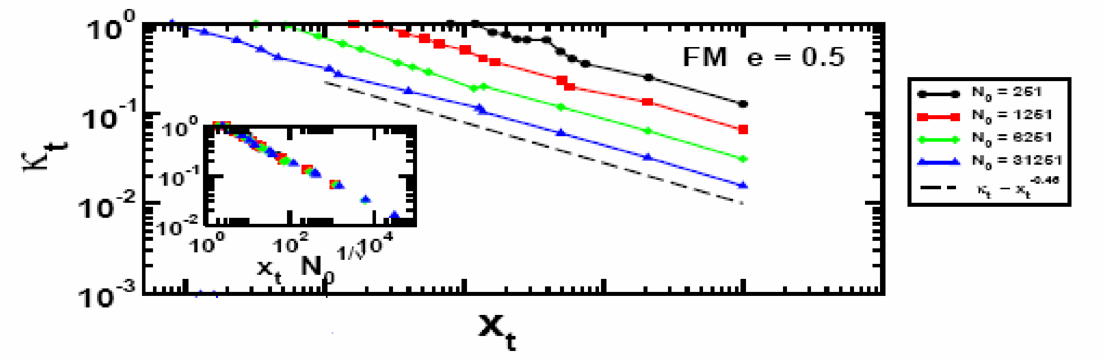

Fig. 6[12] Scaling behavior for a fractal model.

Fig. 6[12] displays the scaling behavior of κ_t for a fractal model, which is self-similar, put forward by C. Song *et al*[14] with hub-hub attraction probability e = 0.5. The critical exponent is $\upsilon = 1$ here and dependent on the box-covering scheme used. Generally, $\upsilon \neq 2$ for self-similar graphs.

Thus, we have obtained two fixed points, self-similarity and non-self-similarity, in our renormalization of the complex networks. Which fixed point is more stable? When additional links are added to the fractal model in Fig. 6 with a small probability $p = 0.05$, the critical exponent changes from $\upsilon = 1$ in Fig. 6 to $\upsilon = 2$ in Fig. 7[12], which indicates that there is a flow from self-similarity to non-self-similarity.

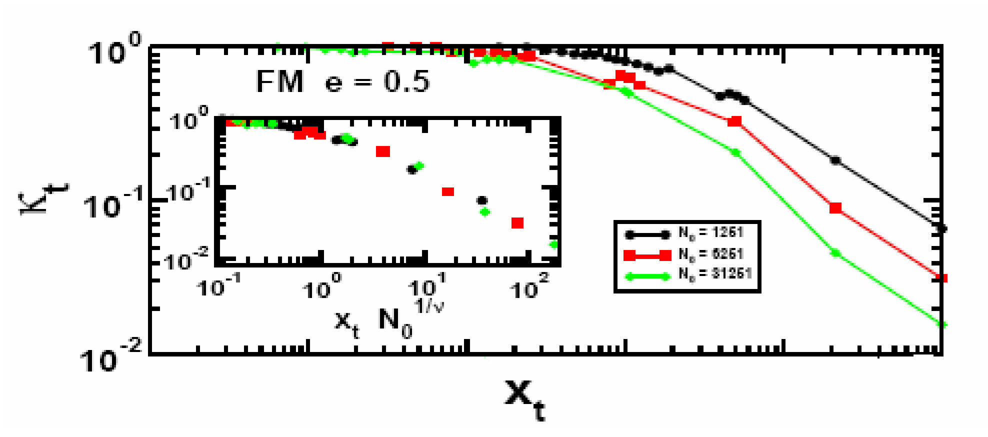

Fig. 7[12] Scaling behavior for a fractal model with small perturbations.

(3) Origin of Self-similarity

The renormalization of the complex networks performed above offers critical exponents which mark obviously whether a network is self-similar or not, and thus classifies each into a universality class. A natural question that follows is the origin of self-similarity.

Naïve hints can be obtained by the study of the correlation profile R(k_1, k_2) which describes the excessive probability, compared to random distribution, for the connection of two nodes with edges k_1 and k_2 respectively. Compared to the profile for Internet, which is non-self-similar, in Fig. 8 (b), Fig. 8 (a) for *E. coli* indicates anticorrelation, i.e. a greater tendency for the connection between larger degree nodes, i.e. hubs, with smaller degree ones for self-similar networks.

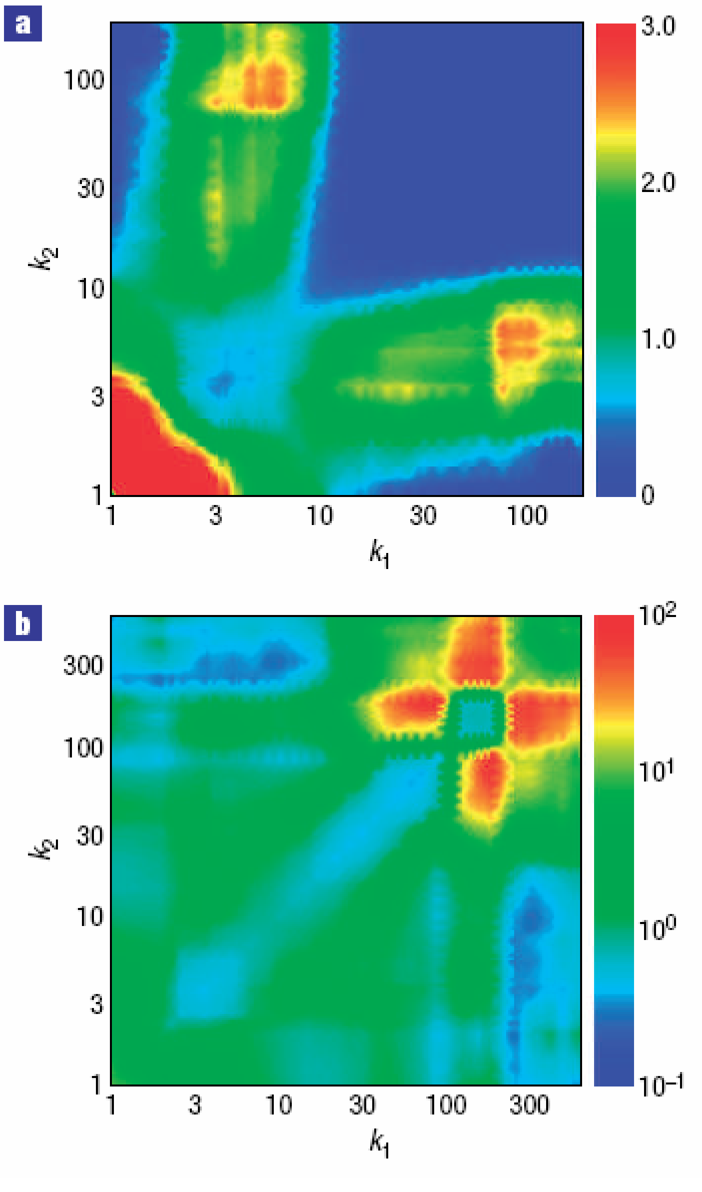

Fig. 8[14] Correlation profile for *E. coli* in (a) and for internet in (b), both with a background profile for World Wide Web.

Thus, C. Song[14] proposed a growth mechanism, inverse to the renormalization procedure, which will yield fractal models. While a box is replaced with a new node in renormalization, every node will be expanded into a box with box size l_B in the growth of the network (Fig. 9). Mode I has a strong hub-hub attraction, where hubs will be connected directly to link the boxes, while Mode II has a strong hub-hub repulsion, where boxes are connected through non-hubs. A network will grow with a hub-hub attraction probability e for Mode I and (1 - e) for Mode II. When e = 1, Mode I is small-world but not self-similar. On the contrary, when e = 0, Mode II is self-similar but not small-world. In the intermediate region 0 < e < 1, a network constructed is both small-world and self-similar (Fig. 10).

To further validate the hub-repulsion origin for self-similarity, the hub connectivity ratio $\mathcal{E}$ is plotted against box size l_B, where $\mathcal{E}$ scales as

$$\mathcal{E}(\ell_B) \sim \ell_B^{-d_e},$$

with d_e describing the anticorrelation strength. In Fig. 11[14], the hub connectivity $\mathcal{E}$ for World Wide Web, which is self-similar, scales with a strong anticorrelation d_e = 1.5, while $\mathcal{E}$ is nearly constant for the non-self-similar Internet, i.e. $d_e \approx 0$. Similar behavior is observed in Fig. 12[14], where the fractal model with e = 0.8 has d_e = 0.66. Thus the resemblance between Fig. 11 and Fig. 12 supports the hub-repulsion mechanism for self-similarity in complex networks.

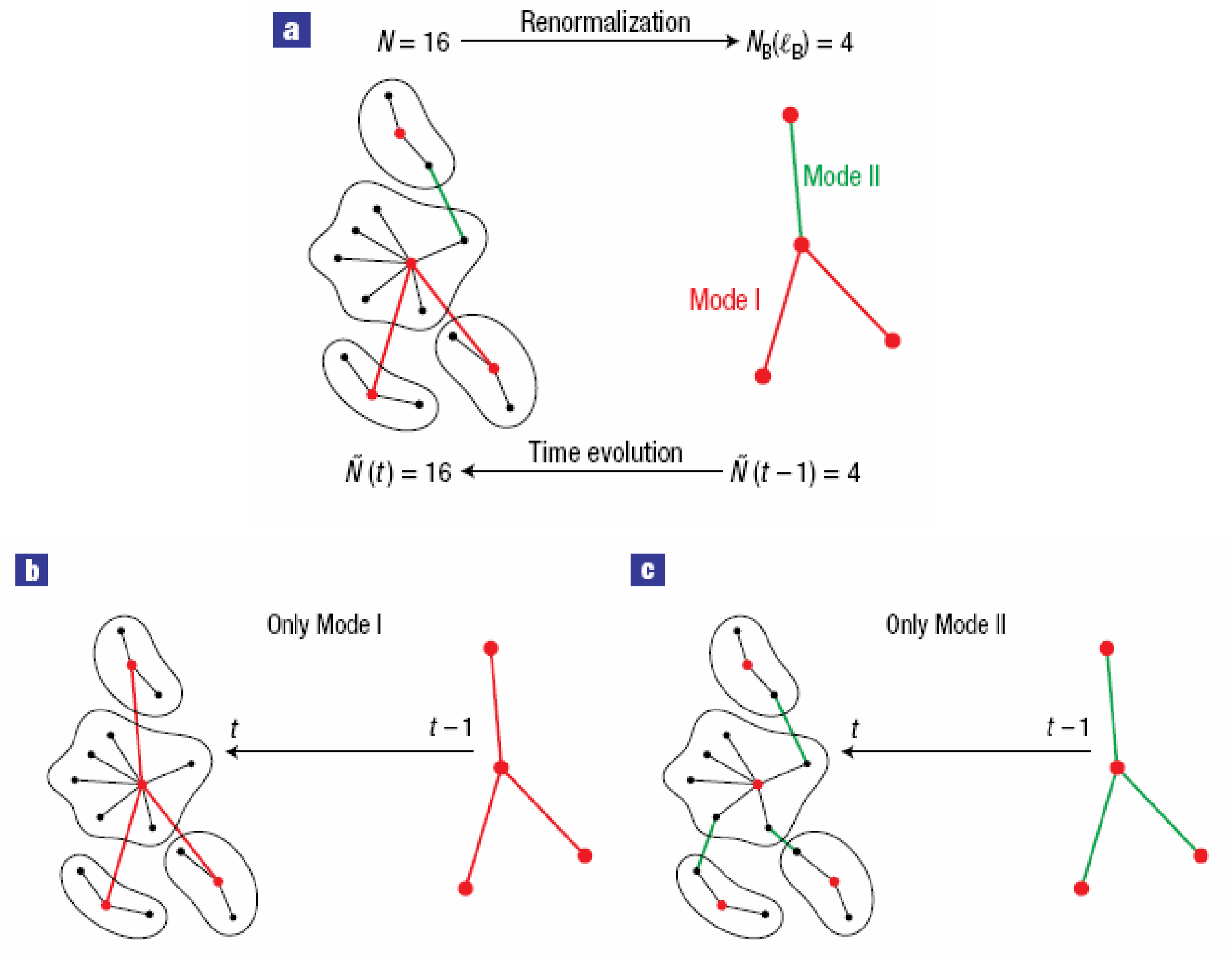

Fig. 9[14] Growth mechanism of a network. (a) Hub-hub attraction probability e. (b) Mode I e = 1. (c) Mode II e = 0.

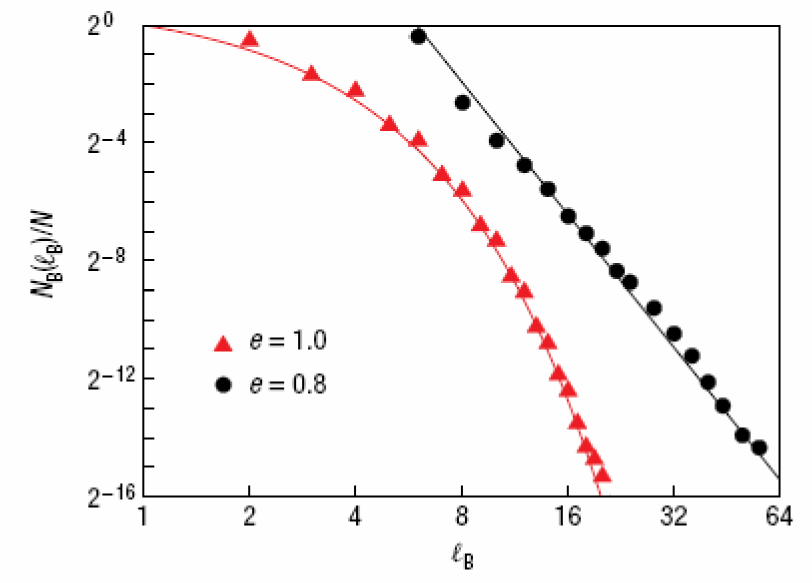

Fig. 10[14] Self-similar network for e = 0.8 and non-self-similar for e = 1.0 (Mode I).

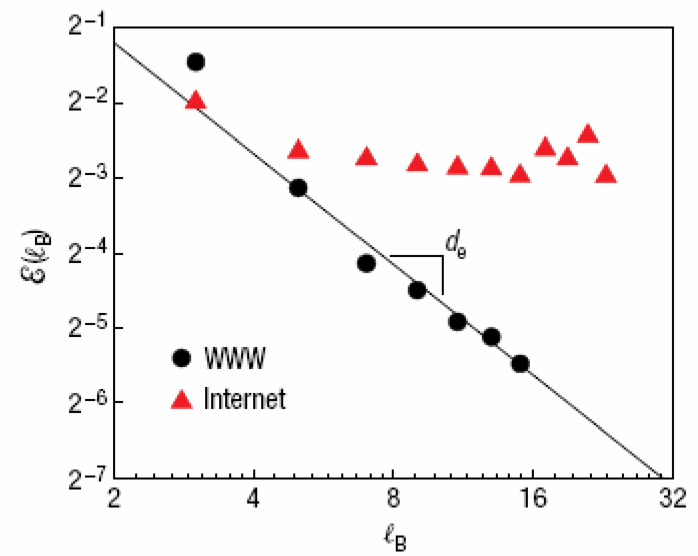

Fig. 11[14] Scaling of $\mathcal{E}$ for World Wide Web and Internet.

(4) Conclusion and Application

Renormalization of complex networks with the box-covering algorithm yields scaling laws and critical exponents distinctive for self-similar and non-self-similar graphs, which may be a good criterion to category universality classes. The self-similar character may be attributed to the hub-hub repulsion in organization. Thus, when some hubs are damaged or removed from the graph, the whole network is better protected and more robust compared to non-self-similar ones[14, 15]. This will account for the widely observed self-similarity in biological networks, which is consistent with the modularity feature in phenotype and genotype, and in functions and morphologies as well. Technological construction, such as a power grid or information resource, may also be designed into a self-similar fashion to give more stable and robust performance.

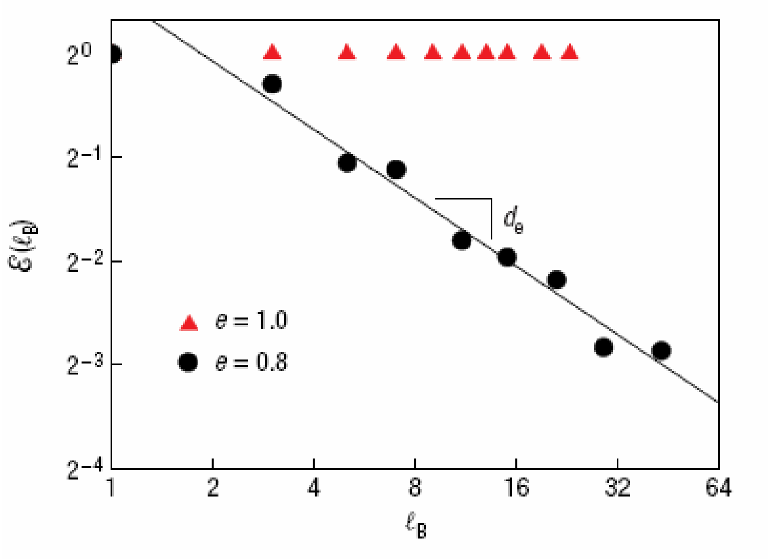

Fig. 12[14] Scaling of $\mathcal{E}$ for Mode I (e = 1.0) and fractal model (e = 0.8).

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