SZNAJD MODEL AND ITS APPLICATION TO POLITICS

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ABSTRACT: Sznajd model of opinion formation in socities has found many applications, primarily in politics. Here, we review the basic model, show that its natural extension to two dimensions exhibits a phase transition, and then focus on its application to proportional elections where our model predicts the exponent for the power law distribution of number of votes obtained by different candidates in Brazilian proportional elections.

1. **Introduction**

The problem that we wish to solve is how to model opinion formation in societies. To this end, we identify the society as our physical system. The individuals and their interactions with others constitute the microscopic scale of our system, whereas the general opinion of the society is the macroscopic observable. In constructing our model, we wish to connect these two scales. Our model should describe a simple protocol on the microscopic level that will enable our system to evolve, where evolution corresponds to opinion formation, resulting in realistic predictions for the general opinion of the society. A society is made up of individuals, so obviously the general opinion of the society will be determined by how individuals form their opinions. In search for an answer to this question, we make the fundamental assumption that opinion of an individual is based on the opinions of its neighbors. This assumption is well supported by various experiments conducted by social psychologists[1], among which the Asch conformity experiment is well known[2].

2. **Original Model**

Description of the model

In search for a simple model of opinion formation, Katarzyna Sznajd-Weron and Josef Sznajd considered a society where each individual time and again has to decide between two choices (A or B)[3]. We note that this setting applies particularly well for referendums, where electorate vote 'Yes' or 'No'. To keep the model simple, it was also assumed that the society in question was a closed society, meaning that from the first timestep the votes are taken until the last timestep, there is no inclusion or exclusion to the electorate; the same people vote over and over again. The concept of a closed society can be interpreted as analogous to an isolated system in physical language. The final assumption that the model had to fit was that the steady states of the model are the three limiting cases:

(a) Each person votes for A
(b) Each person votes for B
(c) Equal number of people vote for A and B

The model consists of individuals, identified as magnets or spins, being ordered on a one dimensional Ising chain. Hence, each individual is regarded as a spin variable S_i where $S_i = 1$ corresponds to voting for A, and $S_i = -1$ corresponds to voting for B. The spins evolve in time according to the following dynamic protocol[4]:

(a) In each timestep, a pair of neighboring spins S_i and S_{i+1}, henceforth which we will refer as a bond, is chosen randomly to interact with their two nearest neighbors S_{i-1} and S_{i+2}.
(b) If $S_i = S_{i+1}$ then $S_{i-1} = S_i$ and $S_{i+2} = S_i$.
(c) If $S_i = -S_{i+1}$ then $S_{i-1} = S_{i+1}$ and $S_{i+2} = S_i$.

We can see that the protocol defines the influence of a bond on the decision of its nearest neighbors, hence in accordance with our fundamental assumption that is given in the introduction. If spins forming the bond point in the same direction, then the nearest neighbors of the bond also point in that direction; whereas if the spins forming the bond point in different directions, then the nearest neighbors point in the opposite direction realtive to them, resulting in an alternating sequence of four spins. Furthermore, it was observed through simulations that the suggested protocol does indeed result in one of the three steady states assumed above, so we may conclude that the model fits all of the assumptions that were in mind previously.

At this point, we pause for a moment to point out an important difference between the Sznajd model and Ising-type models. Before identifying this difference, we emphasize that it is natural to compare this model to Ising-type models since the setting of this model consists of individuals that have two possible states(A or B), ordered in a one dimensional chain. So, the setting of the Sznajd model is identical to the one of Ising. The difference between the two models becomes evident when we consider the interactions among spins. In Ising-type models, the information flows inward. We randomly pick a spin, and see that the behaviour of this spin is governed by the two surrounding spins. Hence, information comes from outside and flows inward to the center, which is the initially picked spin. In contrast to Ising-type models, in the Sznajd model, we randomly pick a bond and see how this bond influences the neighboring spins that are outside of this bond. Hence, information flows outward from the initially picked bond, which is the center, to the surrounding spins.

Simulation and Results

First we define the magnetization of our spin system, which in this case corresponds to the general opinion of the society. We give the definition both in the language of the voting society and in magnetic language:

$$m = \frac{number\ of\ A\ votes - number\ of\ B\ votes}{N} = \frac{1}{N}\sum_i S_i \tag{1}$$

where N is the number of people in the society.

We note that the magnetization defined above is the macroscopic observable of our system, and evidently the important parameter in real life since we are ultimately interested in whether opinion A or opinion B dominates the society. To this end, we must investigate the time evolution of the magnetization. It is clear that after infinite time, the magnetization should evolve to either -1,0 or 1, since the final state of the system is either all spins down, equal number of up and down spins, or all spins up, in accordance with the previous assumptions. In other words, the system should evolve to one of the fixed points.

Following the described protocol above, a Monte Carlo simulation was performed[3] where $N = 1000$ and free boundary conditions apply to the Ising chain. The simulation was started from a totally random initial state, meaning that every spin is assigned -1 or 1, each with probability 1/2. At each timestep, a bond was picked at random, and possible updating of the two nearest neighbors to this bond was performed according to the described protocol. The simulation was iterated until the system was observed to settle in a steady state. It was observed that the final state of the system was one of the three assumed steady states all spins up, all spins down, and equal number of up and down spins, with probabilities 1/4, 1/4, and 1/2 respectively. We note that the probability of all spins up state must be equal to the probability of all spins down state, purely from symmetry considerations. An interesting feature of the steady state with equal number of up and down spins was that the spins alternated orientation in this steady state. In other words, each individual disagreed with its nearest neighbors. We will henceforth call this state the stalemate state, in accordance with popular terminology[3]. In magnetic language, the stalemate state corresponds to an antiferromagnetic state, whereas unanimity in the society corresponds to a ferromagnetic state. From the simulations, it was also observed that the relaxation time, the number of timesteps for the system to reach one of the fixed points, for $N = 1000$ was on the order of 10^4 timesteps. Here we present a graph taken directly from [3].

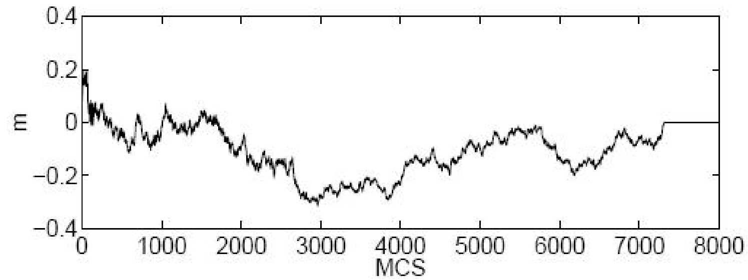

Fig. 1. Time evolution of the magnetization

Figure 1 shows us the time evolution of magnetization in a simulation, where the system finally reaches the stalemate state, reflected by the line segment at $m = 0$. MCS on the figure stands for Monte Carlo steps. Furthermore, we note that this figure is evidence that the system does indeed settle down to one of the assumed steady states, and that the relaxation time is on the order of 10^4 timesteps.

Effect of Initial Conditions

It is natural to expect that the initial conditions have an important effect in determining which one of the three fixed points the sytem evolves to. Moreover, a possible phase transition might be observed at certain initial conditions. In other words, going from one set of initial conditions to another, the final state of the system might change in a way that suggests a phase transition. So, in search for making this dependence on initial conditions more precise, simulations were performed on systems with a fraction c_B of the spins pointing down initially, or equivalently a fraction c_B of the individuals voting for B. In the simulations, initially $c_B * N$ spins were randomly placed to point down, and the rest were taken to be pointing up. We note that from symmetry breaking considerations, if $c_B < 0.5$ then it's more likely that the system will settle down to an all spins up state than to an all spins down state, and vice versa for $c_B > 0.5$. Here we present a plot regarding the effect of initial conditions, taken directly from [3]. Probability of the final state as a function of initial c_B is plotted for each of the three possible final states.

From figure 2, it is interesting to note that in order to end up in an all spins down final state with probability > 0.5, we must ensure that $c_B > 0.7$ initially. We also observe that the system doesn't exhibit a phase transition since the dependence on initial concentration is smooth. Later, we will see that this smooth dependence will be altered in the two dimensional version of the model, and the system will exhibit a phase

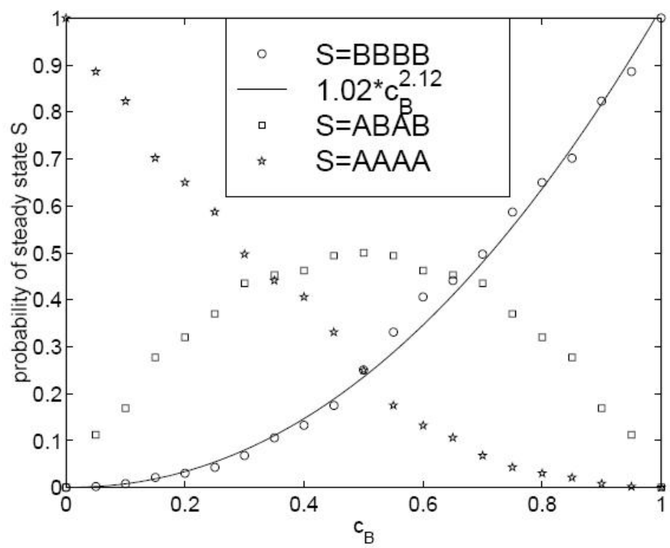

Fig. 2. Dependence of the final state on the initial concentration c_B

transition.

Modification to the Original Model

On the macroscopic level, it seems that the stalemate state as a steady state is rather unrealistic if we think of referendums or other elections where people vote for one of two possible choices. So it's better to modify the original model in a way that accomodates this. On the microscopic level likewise, part c of the above protocol, that is if $S_i = -S_{i+1}$ then $S_{i-1} = S_{i+1}$ and $S_{i+2} = S_i$, seems highly unjustified. Why would an individual oppose the neighboring opinion just because the neighboring opinion doesn't match with its own neighbor? It is more reasonable to assume that the individual keeps its opinion in such a case, rather than opposing its neighbor. So one obvious modification to the original Sznajd model is to eliminate part c of the above protocol[4], that is if $S_i = -S_{i+1}$, nothing happens at that timestep, the system is not altered. It was mentioned in [4] that by eliminating this undesirable rule of the protocol, we also eliminate the unrealistic steady state of stalemate, leaving the system with all spins up state or all spins down state to settle. So we conclude that this modified model without part c is more realistic, hence better, than the original model, both microscopically and macroscopically.

3. **Two Dimensional Model**

Extension of the model to two dimensions

Clearly, assigning four nearest neighbors to an individual rather than only two is a more realistic simplification of a real society. So in search for a more realistic model, two dimensional version of Sznajd model was proposed by Stauffer et al.[5]. In accordance with the dimensionality of the model, six different versions were proposed. Here, we focus on two of the most realistic of the six. We follow the description in [5].

(a) A 2×2 square of four neighbors are chosen at random. If all four spins of the square are aligned in same direction, then all eight neighbors to this square follow the same alignment. Otherwise, the system is not altered, the neighbors are unchanged.

(b) A bond is chosen at random. If the two spins forming the bond are aligned in the same direction, then all six neighbors to this bond follow the same alignment. Otherwise, the system is not altered, the neighbors are unchanged.

Simulation and Results

Following the two rules described above, simulations were performed on a $L \times L$ square lattice[5], where individuals were placed on the lattice as spins in the two dimensional Ising model. The simulations used $L = 101$, helical boundary conditions horizontally, and periodic boundary conditions vertically. As was used before for the original Sznajd model simulation, 'random sequential updating'[5] was used, and the system was iterated until it reached a steady state. It was observed that the system evolved to all spins up or all spins down steady states, each with probability 1/2. We recognize this result as the result for the one dimensional model with the modification that was described in the previous part. We conclude that the two dimensional system evolves to the same fixed points as does the one dimensional system. The people all end up either voting for A or voting for B, making the decision of the society unanimous in the long run.

Initial Conditions and Phase Transition

As we have seen before for the one dimensional model, the effect of initial conditions on the time evolution of the system was investigated for two dimensions[5]. The first rule described above was used to perform the simulations. In consistence with previous definitions, we define the initial concentration of up spins to be c_A where $c_A = 1 - c_B$. Simulations were performed for various initial concentrations c_A, taking 1000 samples for every different c_A, and the dependence of the final state of the system on the initial concentration c_A was plotted on a graph. We present this graph on figure 3, taken directly from [5].

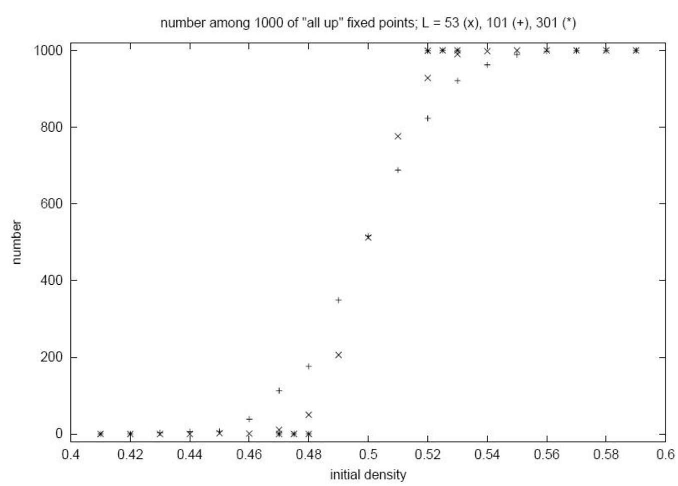

Fig. 3. Phase transition in 2D

It should be noted that the same simulation was performed for three different system sizes to see how the dependence in question changes as the system size is increased. From the step function like dependence on the graph, we conclude that the system exhibits a phase transition at $c_A = 1/2$ since the system almost always settles down to an all spins up state for $c_A > 1/2$, and to an all spins down state for $c_A < 1/2$. Furthermore, it is obvious that as the system size increase, the curve looks more and more like a step function, from which we may conclude that as the system size approaches infinity, analyticity is lost and the notion of a phase transition becomes mathematically precise. We note that this phase transition is not trivial since the dependence of the final state of the system on c_A was smooth in the one dimensional case where the system didn't exhibit any phase transition. In fact, the emergence of a phase transition stands out as the only important difference between the one dimensional model and the two dimensional model.

4. **Application to Politics**

Among the many interesting applications of the Sznajd model of opinion formation to politics, finance, and marketing[4], here we focus on one successful application to proportional elections. Proportional elections, in contrast to single-winner elections, ensure that the representation of political parties/groups in legislature are in proportion to the number of votes they receive from the electorate. This is regarded as a more fair electoral protocol than the single-winner type since in this way all voters are represented in the legislature as opposed to single-winner elections, where the followers of the winning party are represented only.

The Experiment: Brazilian Elections

The experiment for which we wish to fit a theory is the proportional general elections that were held in Brazil in 1998. In these elections, the voters choose state deputies for their states, as well as congressmen for the National Congress. The local elections where state deputies are chosen will be focused on in what follows. Looking at the results from these elections, which is the raw experimental data, Costa-Filho et al.[7] considered the distribution of votes for different candidates in various large states of Brazil, as well as the overall distribution using data for all the states in Brazil. For each state, the number of votes for each candidate was divided by the total number of votes for that state to convert these numbers into fractions of the total. In what follows, $N(v)$ is defined to be the number of candidates that received a fraction v of the total number of votes for that state[7]. The distribution of $N(v)$ constitutes the experimental observation. The elections in the state of Sao Paulo was of particular interest since this state is the largest state in Brazil, and had the most number of candidates running for the state deputy post. In search for a power law behavior, a log-log plot of the voting distribution $N(v)$ was generated for the state of Sao Paulo and for the nationwide voting totals. We note that the distribution for the nationwide voting totals is nothing but the sum of the distributions for different states, before taking the logarithm of the distribution. Here, we present this plot in figure 4, taken directly from [7].

We see that for the state of Sao Paulo, $N(v)$ depends on v in a power-law fashion[7]

$$N(v) \propto v^{-\alpha} \tag{2}$$

with $\alpha = 1.03 \pm 0.03$, valid for about two orders of magnitude. Observing the value of α from figure 5 for the nationwide voting totals, and doing similar plots for other

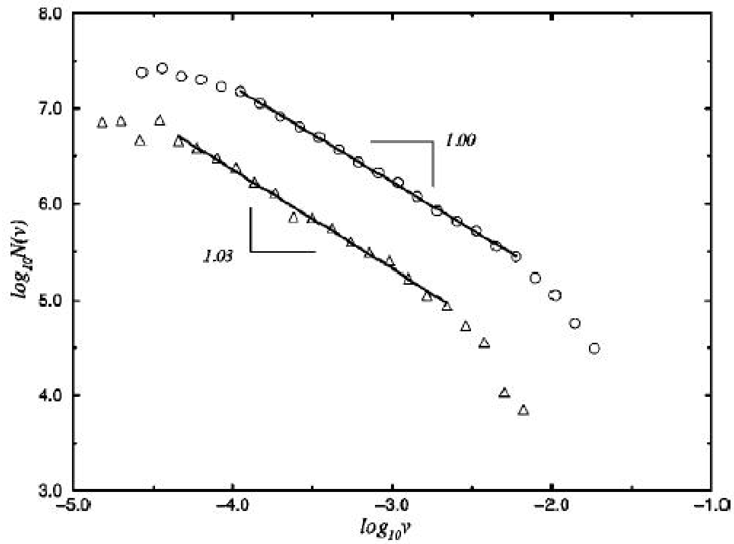

Fig. 4. Distribution of votes from Brazilian elections

sufficiently large states like Sao Paulo, it was concluded that $\alpha \approx 1$ in all cases. Hence it was proposed that the distribution of votes in such proportional elections was scale-invariant; the scaling exponent $\alpha \approx 1$ appeared irrespective of the size of the electorate, whether it be for a state or the entire country. Here we should note that in order to obtain a power law distribution and hence to predict the correct value of the exponent, this experiment should be done on electorate with sufficiently large number of voters and candidates.

Modifying the 2D Model

We wish to write down a theory to fit the experimental observation. To this end, we must develop a model for proportional elections of the sort described above. If we take the two dimensional Snajd model as our starting point, a very obvious modification to the model to account for these kind of elections is to allow more than two opinions for each individual. So if there are N candidates running for the post, then each individual can have N possible opinions. This is an example of a 'many-opinion modification'[4] to the Sznajd model, and also equivalent to extending the Ising model to N-state Potts model.

Now we describe how Bernardes et al. modified the two dimensional Sznajd model in [6]. Naturally, their modified model includes a many-opinion modification. Firstly, the setting is similar as described before: the system is composed of $L \times L$ square lattice of spins, where each spin can be in one of N possible states. Clearly, a state corresponds to voting for one of the N candidates. The candidates are indexed by n, so that a voter who is in state n is voting for nth candidate. The second modification accounts for the influence of the candidate on the voter. This is certainly a necessary and important modification since in real life, the political parties/candidates certainly have influence over the voters. It seems unpromising to try to fit a model that considers only voter-voter relations, and completely disregards party/candidate-voter relations, to real life elections. This modification is introduced in the model by a term

$$P_c(n) = (\frac{n}{N})^2 \quad (3)$$

where $P_c(n)$ stands for the probability of convincing the voter for the nth candidate[6]. Clearly, the dependence of the number of votes on n is monotonic. Number of votes that a candidate obtains increases with increasing n.

In the first stage of the model, the initial condition for the system is determined[6]. Firstly, a random permutation of the sites was generated. Following the order given by this permutation, each site was visited exacly once. At each visit, a candidate n was picked at random among possible N. Then a random number r, where $0 \leq r < 1$, was generated, and compared with P_c. If $r \leq P_c$, then the voter was convinced to vote for n, and another random number q was generated. If $q \leq P_c$, then the voter tried to convince its neighbors to vote for the same candidate in the following way: for each of its four neighbors that vote for n at that timestep, all the six neighbors to this bond were convinced to vote for n. If $q > P_c$, or no neighbors to the originally visited voter are voting for n at that timestep, no neighbors are altered. If $r > P_c$, then nothing was altered, and the next site on the permutation was visited. The first stage ends after the above prtocol is applied to the last site on the permutation. The state of the system after the first stage is identified as the initial condition. We note that both the voter-voter relations and party/candidate-voter relations are in effect in setting up the initial condition.

In the second stage, the second rule which was described above for the two dimensional Sznajd model was used to iterate the system. At each timestep, a bond was randomly chosen on the square. If the members of the bond were in agreement, then all six neighbors would agree; otherwise nothing was altered. We note that only voter-voter relations are in effect in the second stage of the simulation; that is, $P_c(n)$ never comes into play explicitly. The system was iterated for times t such that $1 << t <<$number of timesteps for the system to reach a steady state. We refer to this timescale as transient time following [6]. So, unlike before, the steady states for the system were not investigated, which is in accordance with the purpose of proportional elections that all the voters should be represented in legislature.

The Scaling Exponent Reappears

The simulation was performed for a square lattice with $L = 5001$, which gives 25 million voters, and $N = 2000$, which is the number of candidates. We note that these numbers are comparable to their counterparts for the Sao Paulo state elections considered above. The distribution of votes among candidates was investigated for transient time, and a similar log-log plot as described in the above experiment was generated in search for a power law behavior. Here, we present this plot in figure 5, directly taken from [6]. We note that the number of candidates on the vertical axis was normalized via division by total number of candidates before logarithm was applied.

We see that the middle portion of the curve exhibits power-law character with exponent ≈ -1, in accordance with previous experimental observations from Brazilian proportional elections. So the scaling exponent $\alpha \approx 1$ reappears as a result of the modified Sznajd model.

5. **Conclusion and Further Modifications to Snajd Model**

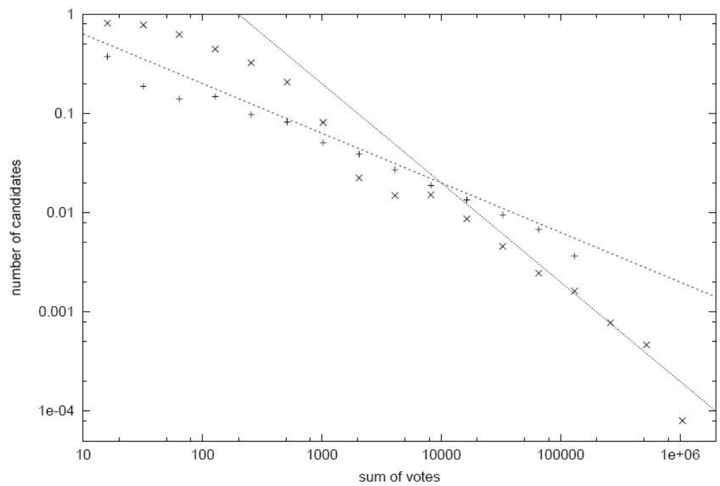

Fig. 5. Distribution of votes from modified Sznajd model simulation

In search for developing a realistic model for the important mechanism of opinion formation in societies, we have started out by considering the original one dimensional Sznajd model[3], which puts the individuals on a one dimensional Ising chain and gives each individual two opinions to choose from. Then, we have looked at the extension of this model to two dimensions[5] as a first step towards realism. We have seen that the two dimensional model exhibits a nontrivial phase transition, which was not present in the one dimensional case. In order to broaden the applicability of the model from referendums or two-candidate elections to multi-candidate elections, we then focused on a particular multi-opinion modification of the model[6], and saw that this modified model predicts correctly the exponent for the power law distribution of votes in Brazilian proportional elections[7]. This new model introduced another major modification to original Sznajd model; it accounted for candidate-voter relations as well as voter-voter relations, whereas the original model only accounted for voter-voter relations.

We believe that the multi-opinion, and candidate-voter relation modifications are necessary and important in rendering the model more realistic, but of course the model could be modified further in pursuit of more realism. One obvious weakness of the model is putting individuals on a two dimensional lattice like Ising spins; as we very well know, people don't live on sites of a lattice. To account for this weakness, the model could be modified to account for long range non-nearest neighbor interactions. Furthermore, we believe that a stochastic term should be introduced to the model to account for the caprice of the voters, as in real life people's caprices are often decisive in directing them towards a decision. The reader should refer to [4] to learn about further studies done on modifying the Sznajd model. In closing, we would like to point out that real life elections are the most reliable experiments to test any kind of modified Sznajd model or other opinion formation models, hence it seems very natural to apply opinion formation models to politics.

6. **References**

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