From Random Walks to Critical Phenomena and Conformal Field Theory

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Abstract

In class, we approached critical phenomena from the view point of correlation functions and discussed renormalization group methods for obtaining values of critical exponents. Here I discuss a parallel idea that studies critical phenomena from properties of random curves which form the domain walls of the critical system. It will be shown that such random curves obey a stochastic differential equation which can be regarded as a conformal mapping. By studying the behavior of these conformal maps, such as the likelihood that a random walk connects two points or encompasses a certain point, one can make connections with quantities in conformal field theory. In addition, it will be shown that the critical exponent can be regarded as the diffusion constant for such random walks. This is a new field which has recently attracted the attention of theoretical physicists and many questions still remain. A general overview is presented here.

1 Introduction

Deep mathematics and physics have had a long love-hate relationship with intermitent renaissances of collaboration. In the area of field theory and string theory, Atiyah and Witten have forged deep relationships between geometry and physics and some of this has been manifested in the study of the fractional quantum Hall effect [1,12]. But despite such interactions, for better or worse, the study of critical phenomena have seen relatively little interaction with exact mathematics. In recent years, however, there has been a developing relationship between the study of Schramm (stochastic) Loewner evolution (SLE) and critical phenomena. Born out of probability theory and the measure theory for random curves in the plane, theoretical physicists have been finding that the mathematicians have devised a framework that not only gives another interpretation of critical phenomena and their associated critical exponents but also a more geometrical picture for the deep ideas of conformal field theory that links it with the study of self avoiding random walks [3,7]. On the one hand, SLE's ask more basic questions: What is the probability that a domain wall in a domain wall forms and encloses an area A? Is there a difference between a random walk at T_c and a random walk at $.5\,T_c$? What's the probability of having a critical cluster span the system size? These are still open questions and instead in this paper, I simply give an overview of what an SLE is, how it relates the study of critical phenomena and predicts critical exponents and some of the insights it gives into conformal field theory. To frame these developments properly, its useful to review historically some of the discoveries in critical phenomena.

1.1 Historical Development of Critical Phenomena

The study of critical phenomena has long captured the attention of physicists and has been a breeding ground of many new ideas in theoretical physicists as they attempted to devise more elaborate tools for characterizing different critical phenomena.

Following Onsager's exact solution for the free energy of the square lattice Ising model, steady progress was made to find exact solutions to many physically relevant integrable models. While many of these solutions gave exact results for critical exponents and critical temperatures, they shed very little light on the general nature of critical systems.

The key insight to critical phenomena came in the 1960's by Wilson, Kadanoff and Widom work the renormalization group. It was realized that in the scaling limit where both the correlation length and all macroscopic length scales are much larger than the length scale associated with microscopic interactions, critical classical systems are equivalent to renormalizable field theories with the same symmetries. Near criticality, correlation functions of the microscopic theory became equivalent to correlation functions of the relevant operators in these quantum field theories [6].

There was then a push to classify all possible renormalizable quantum field theories and to systematically label which classical models went to which suitable renormalizable quantum field theories in two dimensions and to determine all the relevant operators in the quantum field theory. Essentially this was carried to its logical conclusion (in many cases) through conformal field theory which classified renormalizable field theories through something called the central charge c and the scaling dimension of the operator h.

By knowing (c, h) one is able to tabulate all the possible relevant operators in a particular theory [5]. In recent works connections between the central charge and SLE's have been developed [2, 8, 9]. Conformal field theory also revealed a deeper algebraic structure that is present in critical systems. It was shown that correlation functions involving certain operators in this algebra must satisfy a specific differential equation. In conformal field theory, one regards this procedure as inserting an operator (as a pole in the complex plane, representing the continuum critical system) insertion. Once again, SLEs have revealed a more physical interpretation of this procedure: they regard this operator insertion as a boundary condition requiring a random walk to end on it. There also is a notion of the renormalization group where operator insertions can be reduced to the relevant (primary) operators of the theory. Though powerful, the picture is perhaps not as physical and imprecise mathematically. Exactly what are these operators that are being inserted? How should one interpret an operator insertion physically?

Perhaps more alarming is the notion of conformal symmetry. Instead of simple scale (dilation), rotation and translational invariance, the idea of conformal symmetry as described by Polyakov to include *all* analytic functional transformations of the plane needs physical interpretation. What does it mean to change the scales as a smooth function over the entire plane?

1.2 The SLE Point of View

In the stochastic-Loewner evolution picture, the approach is not to invoke mathematically imprecise notions of operators acting in the complex plane, followed by operator insertions in correlation functions and expansions using the Ward-Takahasi identity. Nor does the point of view invoke a renormalization group type of transformation which is mathematically cumbersome.

Instead, the main advantage of the SLE point of view is its generality. It should apply to *any* critical statistical mechanical model in which it is possible to identify non-crossing paths (i.e. domain walls). This approach focuses on the properties of a single curve, conditioned to start at the boundary of a domain, in the background of many critical clusters. This leads to a very specific and physically clear picture of what it means to be conformally invariant [7].

As was shown by Loewner it also turns out that any such curve in the plane which does not cross itself can be described by a dynamical process called a Loewner evolution, in which the curve is imagined to grow continuously [10]. In this picture, instead of looking at the final result, the domain wall, one considers how the curve grew and the likelihood it grew to the final condition.

Now, instead of describing this process directly, Loewner considered the evolution of an analytic function which conformally mapped the region outside the curve into a given domain. A curve of this type turns out to be given completely by a real, continuous function a_t.

Schramm then argued that such a curve *must* be described by a one-dimensional Brownian motion with a *single* parameter, κ, the diffusion constant left unknown. Indeed, the Ising model, Potts models, XY model and self-avoiding random walks have been shown to correspond to particular choices of κ.

With the assumption that SLE describes such a one-dimensional Brownian motion, many properties like the critical exponents have been calculated rigorously. Though interesting, the rigorous mathematics behind the SLE will for the most part be avoided. Rather in this paper, I hope to describe the view of critical systems and conformal field theory as properties of random walks.

2 Random Curves - The Ising Model

Its easiest to think about the Ising model first. The partition function is the usual,

$$Z_{Ising} = \mathrm{Tr}\, \exp\left(\beta J \sum_{r,r'} s(r)s(r')\right) \propto \mathrm{Tr} \prod_{r,r'} (1 + x s(r)s(r')) \tag{1}$$

where $x = \tanh(\beta J)$ and the sum and product are over nearest neighbor sites. At high temperature ($\beta J \ll 1$) one finds a disordered state and at low temperatures ($\beta J \gg 1$) one is in an ordered state: if a spin has a fixed value (say fixed to be up at position r) then even very far away, there is a large probability (not zero) that a spin at a site r' is also up.

The conventional approach to studying critical phenomena is to focus on the behavior of correlation functions of spins. In the scaling limit, one says these spins become local operators in a quantum field theory. The correlation function has the power law behavior $\langle s(r_1)s(r_2)\rangle \sim |r_1 - r_2|^{-2x}$ which is characteristic of a massless QFT (i.e. a conformal field theory). Such a correlation function can be found through field theoretic methods or an exact method like the transfer matrix [6].

But there is another way to think of the Ising model. One can imagine expanding the product (1) out into 2^N terms where N is the total number of edges [4]. On the lattice, one can consider each term $xs(r)s(r')$ appearing in the product as placing a bold link on the lattice connecting sites r and r' (see Figure 1). Each surviving graph is then the union of *non-intersecting* closed loops. Clearly, for a closed path, one finds x^{length} number of factors of $\tanh(\beta J)$ and the partition function can be written as

$$Z = \sum_{\text{possible loops}} x^{\text{length}} \tag{2}$$

When $x = \tanh(\beta J) < 1$ then the dominating terms come from loops with short lengths while if $x \sim 1$, then long loops dominate. This matches with the notion of a disordered state ($\beta J < 1 \Rightarrow x < 1$ and an ordered state ($\beta J \gg 1 \Rightarrow x \sim 1$). Now, in a correlation function,

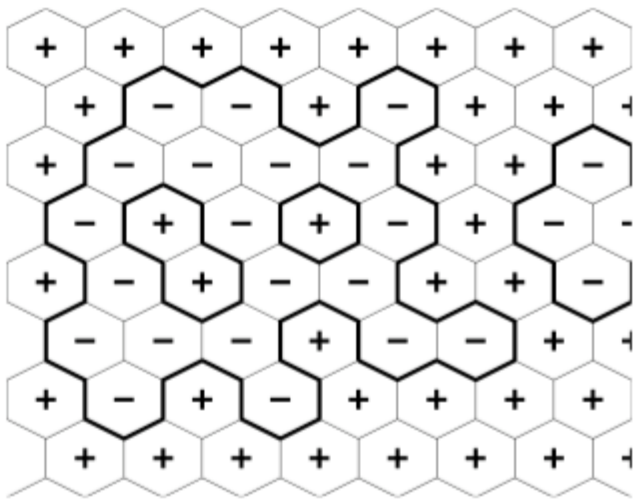

Figure 1: A partition function can be written as a sum over possible closed loops on the lattice. [3].

one considers the quantity,

$$\langle s(r_1)s(r_2)\rangle = \frac{\mathrm{Tr}\ \left(s(r_1)s(r_2)\prod_{r,r'}(1+xs(r)s(r'))\right)}{\prod_{r,r'}(1+xs(r)s(r'))} \tag{3}$$

so one sees that instead of considering closed strings as before, one has an open loop at r_1 and r_2 and one sums over all possible configurations connecting the two points on the lattice (see Figure 2). (One can think of this entire procedure in terms of Grassmann variables and path integrals. By putting Grassmann variables on the lattice sites, one knows that in the path integral vanishes unless each Grassmann variable appears once. Hence the partition function is over closed loops and the correlation function is over open loops with end points at r_1 and r_2 since Grassmann variables at those two point already appear in the trace.) In this language of open loops, one sees that a correlation function essentially amounts to the question, what is the liklihood of a random walk connecting points r_1 and r_2. An SLE is a continuum version of such a curve. Hence, it is plausible that one might obtain the same data describing critical phenomena from the more basic construction of random walks. Many other models, like the Potts models, also can be thought of in terms of the likelihood of domain walls.

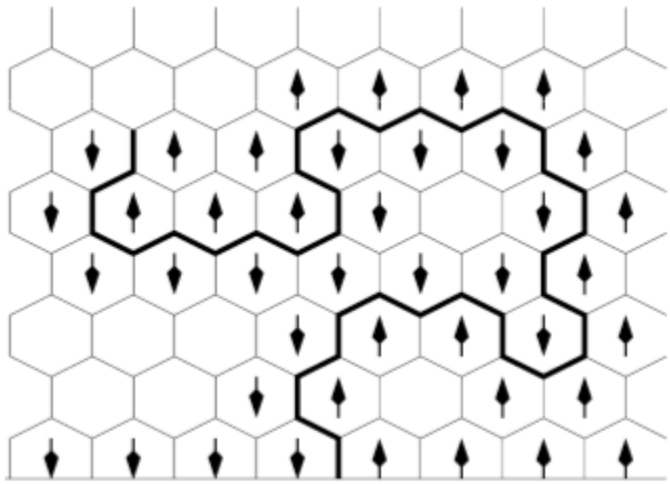

Figure 2: One can think of the correlation function of the probability of walking randomly from one point in the lattice to another [3].

3 Conformal Maps and the Loewner Equation

So far, I have only discussed the the fact that one can also analyze a statistical model by looking at the statistics of a one dimensional random walk on a lattice. The question now is how to describe such a random walk in a useful manner. The key insight for this begins with the Riemann mapping theorem.

One knows that in the complex plane, any analytic function can be thought of as a transformation of one region of the plane, D in the z-plane into another region R in the w-plane through $w = g(z)$. A little more advanced complex analysis tells one that if $g(z)$ is an analytic and univalent (it takes each point on the boundary of D into a unique point. It does not glue points together). In this case, the derivative of $g(z)$ does not vanish in D and takes a the boundary of D into the boundary of R and is said to be conformal. In this way, the study of complex functions connects with the study of curves and regions in the plane.

The conformal functions also have one additional property. If g_a and g_b are functions that map D^a and D^b into the upper half plane respectively. Then, their composition $g_{ab}(z) = g_b(g_a(z))$ maps the part of the region D^{ab} which is contained in D^a into the upper half plane. Hence, the larger region is mapped into a smaller region. Successive iteractions give successively smaller regions (see Figure 3).

3.1 The Loewner Equation

Loewner then considered the evolution of the tip of a random walk as a conformal map that successively mapped the plane into smaller and smaller regions (see Figure 4). The motion

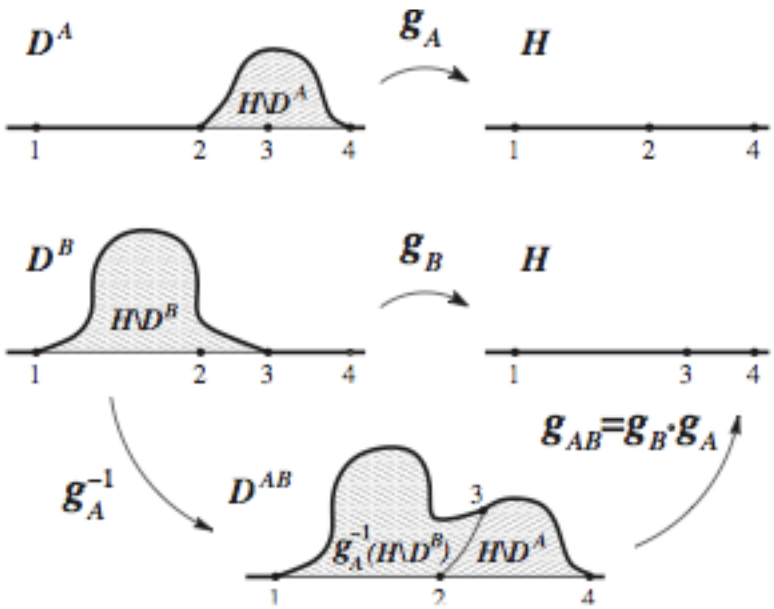

Figure 3: Conformal maps map regions of the complex plane into each other with the map is onto. Successive maps map larger regions into a smaller region [7].

for the is given by [10]

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t} \tag{4}$$

Here a_t is a real driving function. Here, "time" parametrizes a point along the curve. At time zero, $g_{t=0}$ is the identity map which maps the complex plane, H into itself. The boundary curve is the real axis and this is mapped into itself as well. At each subsequent time, $g_t(z)$ defines a new mapping region D_t which maps into a region of H. The boundary of this region is mapped on the real axis [7]. The important properties of g_t come from the composition

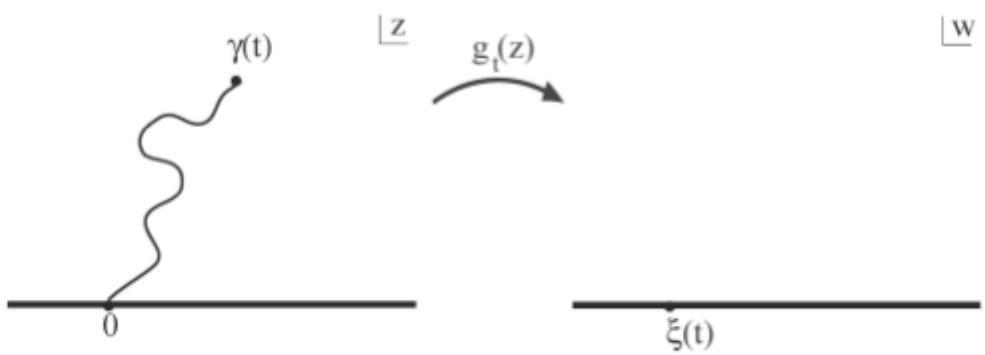

Figure 4: Loewner considers the stochastic evolution of the tip. Each successive "time" step gives a different map of the complex plane into itself. [7]

property. One considers the situation where there are two different driving functions a_t^A and a_t^B which occur in the intervals $[0, t_A]$ and $[0, t_B]$ respectively and the composite forcing function is given by

$$a_t = \begin{cases} a_t^A & \text{when } 0 < t < t_A \\ \\ a_{t-t_A}^B & \text{when } t_A < t < t + t_B \end{cases}$$

Hence in the segment for $0 < t < t_A$ one has a conformal map g_A given by a differential equation with a driving force a_t^A and while in the region $t_A < t < t + t_B$ one has another conformal map g_B. The composition rule then tells us that

$$g_{t_A+t_B} = g_B(g_A(z))$$

gives the evolution of the tip given the past data. An important consequence is that the mapping sets D_t get smaller as t gets larger. Hence if s is greater than t then D_t contains D_s. In a sense, the problem is transformed to a one dimensional random walk along a line with varying step size. One can imagine a series of mappings of segments onto the real line that grow smaller as time goes on.

In the case where a_t, this should be written in terms of differential forms as

$$dg_t = 2\frac{dt}{g_t} - da_t \tag{5}$$

3.2 Critical Phenomena

Originally, Loewner evolution was simply used to describe fractal patterns. The application to critical phenomena came from Schramm who proved that to generated the random walk characteristic of domain walls of the critical system, the driving function *must* be a geometric Brownian motion. Namely,

$$da_t = \sqrt{\kappa} dt \tag{6}$$

4 SLE's and Critical Exponents

To relate the data about random walks to critical exponents, one needs to map the SLE from the plane parameterized by z onto the cylinder with coordinates w through the conformal mapping, $z = e^{-w}$. The Loewner equation becomes

$$\frac{dg_t(z)}{dt} = -g_t(z)\frac{g_t(z) + e^{i\theta_t}}{g_t(z) - e^{i\theta_t}} \tag{7}$$

One can argue again that the variable θ_t for a critical system must be a geometric Brownian motion. Its not immediately obvious how the cylindrical SLE is related to the planar SLE, but it has been shown that if a SLE on the plane is conditioned at $t = 0$ to begin at the origin and hit the boundary at $t = t_1$, then on the cylinder, the SLE is conditioned to begin at the origin at θ_0 and end at angle $e^{i\theta_{t_1}}$. This implies that the planar SLE and the cylindrical SLE have the same κ [3].

Now, the winding angle at time t is simply given by $\theta_t - \theta_0$. Hence, since θ_t is given by a geometric Brownian motion, the winding angle should be normally distributed with variance κt.

For a critical system, one often asks for the correlation function

$$\langle \mathcal{O}(r_1)\mathcal{O}(r_2)\rangle \sim |r_1 - r_2|^{-2x_{\mathcal{O}}}$$

The operators are thought of as distributed normally with mean 0 (for instance the Ising model, $\langle s(r)\rangle = 0$). Hence the correlation function, is the variance for an operator $\mathcal{O}$ *[paper]*. By mapping the plane onto the cylinder of length L and circumference ℓ, the correlation function goes as $e^{-2\pi x_{\mathcal{O}} L/\ell}$ so that the variance of the winding angles goes as $x_{\mathcal{O}} L$. In the SLE, one parameterized the length with the variable t so one identifies $L \sim t$. One then finds that κ is directly identifiable with the critical exponent for a critical theory.

However, this is a tentative correspondence. In truth, it has only been shown to be true for the $O(n)$ models where the relevant operators are $e^{i\phi(r)}$. Proofs are still necessary, but people in the field suspect that the similar proofs exist and that following should be true [3].

1. $\kappa = -2$: loop erased random walks (proven)
2. $\kappa = 8/3$: self-avoiding random walks (unproven)
3. $\kappa = 3$: cluster boundaries of the Ising model (unproven)
4. $\kappa = 4$: double dimer models (unproven). Kosterlitz-Thouless (proven), Gaussian random field (proven) and harmonic explorer (proven)
5. $\kappa = 6$: cluster boundaries of percolation (proven)
6. $\kappa = 8$: dense phase of self-avoiding random walks and boundaries of uniform spanning trees (proven)

5 Schramm's Formula

By thinking about the nature of the domain walls, SLE gives a frame work to answer questions not readily answerable by only considering correlation functions. One such question (which will later turn out to be related to conformal field theories) is the question: given a curve connecting two points r_1 and r_2 on the boundary of a domain D [11].

Let the probability that the curve passes to the left of the point ξ be $P(\xi, \bar{\xi}; a_0)$. $\bar{\xi}$ is included just as a reminder that the function does not have to be holomorphic. Now the Loewner equation is a conformal mapping of the plane onto itself. More importantly, the equation (4) will map the curve onto its image. At the same time it also maps the point ξ to $\xi' = g_t(\xi) = \xi + 2dt/(\xi - a_0)$. It is a simple application of (4) given the intial condition that g_0 is the point ξ. Now, since this is a conformal mapping, Schramm realized that if the curve lies to the left of ξ, then so its image must lie to the left of ξ'. Hence the probabilities must be the same:

$$P(\xi, \bar{\xi}; a_0) = \left\langle P(\xi + 2dt/(\xi - a_0), \bar{\xi} + 2/dt(\bar{\xi} - a_0); a_0 + \sqrt{\kappa} dB_t) \right\rangle$$

where the average is over all realizations of the Brownian motion dB_t up to time dt. Now, Taylor expanding and remembering that for a mean zero distribution $\langle dB_t \rangle = 0$ and $\langle (dB_t)^2 \rangle = dt$ and setting the coefficient of $dt = 0$ since the average is not a random variable, one finds

$$\left(\frac{2}{\xi - a_0} \frac{\partial}{\partial \xi} + \frac{2}{\bar{\xi} - a_0} \frac{\partial}{\partial \bar{\xi}} + \frac{\kappa}{2} \frac{\partial^2}{\partial a_0^2} \right) P(\xi, \bar{\xi}; a_0) = 0 \tag{8}$$

This constitutes Schramm's equation and is a typical differential equation that conditional probabilities satisfy in stochastic differential equations. It turns out that such a differential equation also applies to correlation functions in conformal field theory when one considers primary operator insertions. Physically, this gives the picture that, by inserting the primary operators, one is providing some boundary condition on the domain wall (i.e. if I fix a spin up, then the curve must form a domain wall around a spin up cluster in the critical system).

6 SLE's and Conformal Field Theory

Another object one can consider is the effect of inserting boundary operators into the critical system. In these types of problems in conformal field theory, one thinks of inserting an operator at the boundary. This operator is known to create a domain wall in the lattice that connects it to infinity. For instance, one can imagine fixing one spin in the Ising model at the boundary to be up and watching how the domain wall forms from this point.

Such a question is easily considered in SLE's. The existence of such a curve fixes a set of field configurations ψ. I can write the state of the curve then as a path integral

$$|\gamma_t\rangle = \int [d\psi'_\Gamma] \int_{\psi_\Gamma = \psi'_\Gamma; \gamma} [d\psi] e^{-S[\psi]} \, |\psi'_\Gamma\rangle$$

All this says is that I integrate over field configurations consistent with the domain wall/random walk, γ. Integrating over all possible realizations of the curve γ defines a state of the system

$$|h\rangle = |h_t\rangle = \int d\mu(\gamma_t) \, |\gamma_t\rangle$$

Now the measure on the curve is dependent on a_t in the Loewner evolution through a sequence of iterated maps $dg_t = 2dt/g_t - da_t$. For a geometric Brownian motion, Cardy shows that this relations between the measure relates the value of the highest weight and the central charge to the diffusion constant

$$\begin{aligned} h &= \frac{6-\kappa}{2\kappa} \\ c &= \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \end{aligned} \tag{9}$$

Hence, one finds that different boundary operators in conformal field theory will generate domain walls that have differing diffusion constants [2,3]

7 Conclusion

Some of the most exciting ideas in theoretical physics are those linking algebraic expressions with geometrical interpretations. The geometric meaning of the equations of general relativity, ideas of parallel transport on "surfaces" in Yang-Mill's description of gauge fields and a knot theory description of fractional statistics are just some examples of these. SLE's though a relatively new field, offers a geometric picture for the quantities in conformal field theory. It seeks to describe the patterns of paths of Brownian motion, the forms of percolating clusters, the shapes of snowflakes and phase boundaries or the growth patterns of dendrites and only now are people beginning to relate these geometric objects to the algebraic objects in conformal field theory. Though highly mathematical, it offers an interesting interpretation of quantities in conformal field. Abstract operator insertions or boundary conformal operators are reinterpreted in terms of random walks on the plane. However, much is still unknown in this field, but hopefully a general picture of this field has been presented.

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