Fermi Liquid and BCS Phase Transition

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Abstract

Landau fermi liquid theory is introduced as a successful theory describing the low energy properties of most fermi systems. Besides the usual argument based on the calculation of the life time of the quasiparticles, a renormalization group calculation, of Wilson's shell integration in momentum space, is implemented to justify the validity of fermi liquid theory and shows that BCS phase transition is the only singularity in a homogeneous fermi system.

1 Introduction

Fermions and bosons are the building bricks of the matter around us. The representative of fermions is the electron, whose behavior determines various respects of most solid state systems. Fermions obey Fermi-Dirac statistics. Pauli's exclusive principle says that the ground state of a free fermion system is simply constructed by stacking all the fermions from bottom up to the fermi surface, though the similar problem for interacting fermion systems is totally non-trivial. For example, in ordinary metals, the coulomb interaction energy between electrons usually dominates the kinetic energy, which even makes perturbation theory unapplicable.

In 1956 Landau [1] proposed the fermi liquid theory, which turns out to be a quite successful theory to study the low temperature physics of interacting fermions. The theory postulates that the thermodynamics of an (even strong) interacting fermion system at low energy scale can be duplicated by studying a system of , called quasiparticles, where usual perturbation theory can be employed to calculate all kinds of relevant observables. The physical picture is like this: Suppose an extra electron is placed into the metal. It will immediately get into the interaction with all the other electrons. It is true that the interaction between these bare electrons can be overwhelming the kinetic energy. However, under some circumstances, if in some clever way we can combine the bare electron and partial media surrounding it as a new entity, which we call quasiparticle, the residue interaction between these quasiparticles becomes weak, which enables the use of usual techniques. By the construction, the quasiparticles are still fermions and their density is the same as the underlying bare fermions.

The fermi liquid theory is always justified in a perturbation way [2] that when the quasiparticle is arbitrarily close to the fermi surface, its life time becomes arbitrarily long. It is due to the limitation of the phase space. Though capable of explaining a lot of experiments of interacting fermions, however, the fermi liquid theory can not be applied to a system in the superconductor phase. It is the purpose of this paper, from the modern point of view of the renormalization group theory, to see the validity of the fermi liquid theory and when it is broken down.

2 Fermi Liquid Theory

Landau postulated that at low energy scale the excitation spectrum of the bare interacting fermion system is the same as that of a system composed of weak interacting quasiparticles and the energy of the bare fermion system E is a functional of the distribution of the quasiparticles $n_{\mathbf{k}}$, where $\mathbf{k}$ is the index designating the single particle states got in a free fermion system (for simplicity, let us first neglect the degrees of freedom of spin). The energy spectrum of the quasiparticles are defined by the functional derivative of $E(\{n_{\mathbf{q}}\})$ with respect to $n_{\mathbf{k}}$ as

$$\epsilon_{\mathbf{k}} = \frac{\delta E(\{n_{\mathbf{q}}\})}{\delta n_{\mathbf{k}}}. \tag{1}$$

When k is approaching the fermi momentum k_F, to lowest order, $\epsilon_{\mathbf{k}} - \epsilon_{k_F} = \mathbf{v_F} \cdot (\mathbf{k} - \mathbf{k_F})$, where v_F is the fermi velocity at the fermi surface.

And further the Landau functions introduced to describing the interacting between quasiparticles are defined by

$$\frac{1}{V} f(\mathbf{k}, \mathbf{k}') = \frac{\partial E(\{n_{\mathbf{q}}\})}{\delta n_{\mathbf{k}} \delta n_{\mathbf{k}'}}, \tag{2}$$

where V is the volume of the system. Thus they are symmetric

$$f(\mathbf{k}, \mathbf{k}') = f(\mathbf{k}', \mathbf{k}). \tag{3}$$

The Landau functions $f(\mathbf{k}, \mathbf{k}')$ may be calculated from the microscopic model which is faithful to the bare fermion systems or extracted from experimental data.

Provided that the changing the distribution of the quasiparticles is small and orders higher than quadratic are negligible, the change of the energy of the whole system corresponding is given by

$$\delta E = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{k}'} f(\mathbf{k}, \mathbf{k}') \delta n_{\mathbf{k}} \delta n_{\mathbf{k}'}. \tag{4}$$

Since the quasiparticles are fermion, they also obey Fermi-Dirac statistics and the distribution function at finite temperature is given by the usual form [3]

$$n_{\mathbf{k}} = \frac{1}{e^{\epsilon_{\mathbf{k}} - \mu} + 1}. \tag{5}$$

Note that $\epsilon_{\mathbf{k}}$ is a functional of $n_{\mathbf{k}}$. At zero temperature $n_{\mathbf{k}} = \theta(|\mathbf{k}| - k_F)$ with k_F.

In the fermi liquid theory the quasiparticles behave quite like free particles. This is compatible with the fact that at low temperatures the heat capacity part contributed by the electrons in a lot metals is linear with respect to the temperature. However, to make the quasiparticle concept meaningful, its life time must be long enough. It is usually put in the following diagrammatic language [2]: The life time of a quasiparticle slightly above the fermi surface is mainly governed by its interaction with the quasiparticles below the fermi surface. The perturbation calculation gives

$$\frac{1}{\tau} = \frac{1}{2}\sum_{abB} |\langle \alpha B|v|ab\rangle|^2 2\pi\delta(\epsilon_\alpha + \epsilon_B - \epsilon_a - \epsilon_b), \tag{6}$$

where α designates the state of quasiparticle we study and $\epsilon_\alpha > 0$ (all the energy is defined with respect to the fermi surface). $\epsilon_B(< 0)$ is the energy of the initial quasiparticle in the fermi sea and $\epsilon_a(> 0)$ and $\epsilon_b(> 0)$ are the energies of the final quasiparticles after the scattering. $|\langle \alpha B|v|ab\rangle|$ is the relevant matrix element. τ is the life time of the quasiparticle. When ϵ_α close to the fermi surface, the energy conservation condition, $\epsilon_\alpha = |\epsilon_a| + |\epsilon_b| + |\epsilon_B|$, puts strong limit on the phase space available to the process. Suppose the matrix element is upper bounded by the value $v_{\max}$. A quick estimate gives

$$\begin{aligned}\frac{1}{\tau} \leq &\pi v_{\max}^2 \int_0^\infty d\epsilon_a \int_0^\infty d\epsilon_b \int_0^{-\infty} d\epsilon_B\ \rho(\epsilon_a)\rho(\epsilon_b)\rho(\epsilon_B)\delta(\epsilon_\alpha + \epsilon_B - \epsilon_a - \epsilon_b)\\ =&\pi v_{\max}^2 \int_0^{\epsilon_\alpha} d\epsilon_a \int_0^{\epsilon_\alpha} d\epsilon_b\ \rho(\epsilon_a)\rho(\epsilon_b)\rho(\epsilon_a + \epsilon_b - \epsilon_\alpha)\\ \leq&\pi v_{\max}^2 \rho_{\max}^3 \epsilon_\alpha^2,\end{aligned} \tag{7}$$

where $\rho_{\max}$ is the upper bound of the density of the states around the fermi surface. So $\tau\epsilon_\alpha$ goes to zero when ϵ_α goes to zero. This justifies the concept of quasiparticle.

The fermi liquid theory is a phenomenological or effective field theory. Its power lies in its prediction of various properties of fermion systems with the input of a few phenomenological parameters. It gives results agreeable with experiments. Its validity depends on the condition that when the interaction between the fermions are adiabatically turned on, no bound states are formed and the low energy physics is only happening around the fermi surface. But we know at sufficient low temperatures some metals exhibit transition to superconductors, which is successfully explained by the BCS theory. It is instructive to see, from the modern point of view of the renormalization group theory, why the fermi liquid theory is so successful and how the BCS transition emerges.

3 Renormalization Group Approach

Wilson's series work on renormalization group [4] does not only place the foundation of modern phase transition theory, but also sheds new light on the understanding of field theory. In physics only (at least relatively) low energy and long wavelength phenomena are studied, which is obviously governed by the short distance physics. However, it turns out that without the every detail of the short distance physics, the low energy and long wavelength phenomena can be captured in an effective field theory, in which only finite number of parameters are needed. Of course, these parameters can not be determined at the given level of the description of the problem. They can be either calculated from a model at lower levels or matched by experimental data. Since it is impossible to acquire full knowledge at any short distance, the first way is not so helpful. From this point of view, any field theory must be phenomenological. The crucial point to construct an effective field theory is to note what kind of parameters need to be introduced, and, what is more important, to identify the relevant degrees of freedom. So the applicability of any effective field theory to any realistic system is impossible to be justified priori to any theoretical calculation and its comparison with experiment results.

The reason why the full detail of the short distance physics is not needed for low energy and long wavelength phenomena is that starting from a microscopic model, under the renormalization group transformation to large length scales, all the irrelevant interactions will not appear in the final coarse-grained Hamiltonian. There are various ways to construct renormalization group transformations. In this paper, we implement Wilson's momentum shell integration [5]. The idea is that we first take a model with momentum cutoff Λ; we integrate out the degrees of freedom lying in the region from $b\Lambda$ to Λ, where b is slightly smaller than 1; after the integration, the new cutoff becomes $b\Lambda$ and finally we rescale the momentum from $b\Lambda$ back to Λ; by doing so, all the parameters will be different from the original ones and this gives the renormalization group transformation.

Recently the Landau theory of fermi liquid started to be examined within the framework of the renormalization group theory [6, 7]. Our calculation will basically follow the treatment in these reviews.

The subject of the Landau theory of fermi liquid is not the bare fermions, instead, is the fermionic quasiparticles. We consider a fermion system with spatial rotational symmetry in three dimensional space at zero temperature. The fermi sea must be a sphere. Since all the interesting low energy physics happens around the fermi surface we decompose the momentum as $\mathbf{q} = \mathbf{k}+\mathbf{l}$, where $\mathbf{k}$ lies on the fermi surface and $\mathbf{l}$ is orthogonal to it. The free-field action of the quasiparticles around the fermi surface with a cutoff Λ is

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{dl}{2\pi} \big\{ \bar{\psi}(\omega, \mathbf{k}, l)(i\omega - v_F l)\psi(\omega, \mathbf{k}, l) \big\}, \tag{8}$$

where the spectrum $\epsilon(\mathbf{k}+\mathbf{l}) - \epsilon_F$ has been approximated as $v_F l$ provided $l << k_F$. Now if we scale the cutoff Λ to $b\Lambda$ with $0 < b < 1$, we have

$$l \rightarrow bl, \ \omega \rightarrow b\omega. \tag{9}$$

In order to make the free-field action invariant, ψ is required to scale as $\psi/\sqrt{b}$.

Next we investigate the response of all kinds of perturbation to the free-field action which are compatible with the symmetries of the problem to the scale manipulation, which is equivalent to the tree level calculation of the renormalization group transformation. Since the quadratic term of ψ simply defines the fermi energy, the first interesting case is the quartic interaction of the form

$$\delta S_4 = \frac{1}{2!2!} \int_{\mathbf{k},l,\omega} \bar{\psi}(4)\bar{\psi}(3)\psi(2)\psi(1)u(4,3,2,1), \tag{10}$$

where

$$\psi(i) = \psi(\mathbf{k}_i, \omega_i, l_i), \tag{11}$$

$$\int_{\mathbf{k},l,\omega} = \Big[\prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int \frac{d\mathbf{k}_i}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{dl_i}{2\pi}\Big] \theta(\Lambda - |l_4|). \tag{12}$$

The step function in the integral plays a crucial role. From the conservation of the momentum, we get

$$l_4 = |(k_F + l_1)\mathbf{\Omega}_1 + (k_F + l_2)\mathbf{\Omega}_2 + (k_F + l_3)\mathbf{\Omega}_3| - k_F, \tag{13}$$

here $\mathbf{\Omega}_i$ is a unit vector in the direction of $\mathbf{q}_i$ and $\mathbf{k}_i = k_F \mathbf{\Omega}_i$. If now we carry out the scale scheme directly, we find

$$\theta(\Lambda - |l_4(l_1, l_2, l_3, k_F)|) \to \theta(\Lambda - |l'_4(l'_1, l'_2, l'_3, k_F/b)|), \tag{14}$$

the fermi momentum changes. However in our theory we want the fermi momentum to be invariant since it is determined by the density of the underlying bare fermions. This technique difficulty could be overcome by using a smooth cutoff for l_4,

$$\theta(\Lambda - |l_4|) \to e^{-|l_4|/\Lambda}. \tag{15}$$

Now we define $\mathbf{\Delta} = \mathbf{\Omega_1} + \mathbf{\Omega_2} - \mathbf{\Omega_3}$ and write

$$l_4 = |k_F \mathbf{\Delta} + l_1 \mathbf{\Omega}_1 + l_2 \mathbf{\Omega}_2 - l_3 \mathbf{\Omega}_3| - k_F. \tag{16}$$

We will only keep the Δ term in l_4 and neglect $O(l)$ since for $|l_4| < \Lambda$, the latter part is always much smaller than the former. Then the scale scheme transforms the quartic term as

$$\begin{aligned}
&\prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int \frac{d\mathbf{k}_i}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{dl_i}{2\pi}\; e^{-(k_F/\Lambda)||\mathbf{\Delta}|-1|} u(\mathbf{k}, l, \omega) \bar{\psi}\bar{\psi}\psi\psi \\
&\to \prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega'_i}{2\pi} \int \frac{d\mathbf{k}_i}{(2\pi)^2} \int_{-\Lambda}^{\Lambda} \frac{dl'_i}{2\pi}\; e^{-(k_F/(b\Lambda))||\mathbf{\Delta}|-1|} u(\mathbf{k}, bl', b\omega') \bar{\psi}\bar{\psi}\psi\psi.
\end{aligned} \tag{17}$$

We rewrite

$$e^{-(k_F/(b\Lambda))||\Delta|-1|} = e^{-(k_F/\Lambda)||\Delta|-1|} e^{-((1/b-1)k_F/\Lambda)||\Delta|-1|}, \tag{18}$$

so that the measures before and after have the same factor $e^{-(k_F/\Lambda)||\Delta|-1|}$. By comparing the coefficient, we can read off

$$u'(\mathbf{k}, l', \omega') = e^{-((1/b-1)k_F/\Lambda)||\Delta|-1|} u(\mathbf{k}, bl', b\omega'). \tag{19}$$

So we can conclude that at present (tree) level calculation, the only couplings which survive the renormalization group transformation must fulfill the condition

$$|\mathbf{\Delta}| = |\mathbf{\Omega}_1 + \mathbf{\Omega}_2 - \mathbf{\Omega}_3| = 1. \tag{20}$$

In three dimensions it is equivalent to say $\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2 = \mathbf{\Omega}_3 \cdot \mathbf{\Omega}_4$.

4 Discussion

As shown above the quartic couplings with the condition $\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2 = \mathbf{\Omega}_3 \cdot \mathbf{\Omega}_4$ are marginal and the others are irrelevant at tree level. The condition $\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2 = \mathbf{\Omega}_3 \cdot \mathbf{\Omega}_4$ is the direct result of the factor $\theta(\Lambda - |l_4|)$ in Eq. (12), whose effect can be understood in a geometric way [7]. When Λ/k_F goes to zero and the shell of the momentum space of interest becomes thinner and thinner, while the initial momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ of the two quasiparticles taking part in the scattering process lie in the momentum shell, the momenta of the resultant quasiparticles $\mathbf{q}_3$ and $\mathbf{q}_4$, when required to be within the momentum shell as well, must have the same mutual angle between them as $\mathbf{q}_1$ and $\mathbf{q}_2$ do. This is essentially the same as the argument of the limit on the phase space put in the diagrammatic language. The couplings with $\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2 = \mathbf{\Omega}_3 \cdot \mathbf{\Omega}_4$ but $\mathbf{\Omega}_1 \neq \mathbf{\Omega}_2$, denoted by u_f, are responsible for the forward scattering, and proportional to the Landau functions $f(\mathbf{k}, \mathbf{k}')$. The special case is the coupling, denoted by u_z, with $\mathbf{\Omega}_1 = \mathbf{\Omega}_2$ and $\mathbf{\Omega}_3 = \mathbf{\Omega}_4$. The conclusion that both u_f and u_z are marginal is only true at the tree level. Further, one loop calculation [7] gives u_f still marginal, but u_z marginal relevant for $u_z < 0$ and marginal irrelevant for $u_z > 0$. So at zero temperature, the fix point of the general model of fermionic quasiparticles is determined by u_f and u_z. When both $u_f > 0$ and $u_z > 0$, the Landau fermi liquid theory is valid. If $u_z < 0$, another BCS fix point will appear, which means BCS state should be true ground state if any attractive interaction exists. However, at finite temperatures, the BCS phase may be broken due to thermal fluctuation.

To summarize, we have introduced the Landau theory of fermi liquid and assessed its validity within the frame work of renormalization group theory up to one loop calculation. It is shown that all the forward scattering couplings are marginal, which justifies the fermi liquid theory. And moreover, if the interaction is attractive, BCS state becomes the true ground state.

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