Ising model as a model of multi-agent based financial market

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The real financial market reveal some statistical properties such as fat tail, scaling and volatility cluster. There are three main elements for a financial market: agent dynamics ,agent-price mechanism and price dynamics. The agent dynamics interplay the price dynamics by a agent-price mechanism. A model of financial market based on multi-agent is introduced. Ising model, a statistical model is found to share the similar structure with agent dynamics for this model. Based a modified Ising model, we are able to simulate the agent dynamics thus the price dynamics which show the statistical property of the real financial market.

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1. INTRODUCTION

Over the last fifty years, theory of finance gain great success on both the macroscopic level and microscopic level [1]. As the price theory is the center theory for the commodity. How to determine the stock prices is the fundamental problem in the financial market. At macroscopic level,a basic assumption of most of the theory of the stock prices is the Efficient Markets Hypothesis (EMH) that roughly speaking, all the information affecting the price will mediately reflected in the prices. Or in anther precise but not restrict way, the stock prices are random walks (We will see why this statement is not restrict). This implies that the record of the past stock prices cannot predict the future of the stock prices remembering that a random walker has no memory of his past. So a technique analysis (TA) of stock prices is useless. And this is contradict the behaviors in Wall street, people use technique analysis to analyze the stock prices and make prediction to make money. Another defect of EMH is that it is not a theory that can be tested in real financial market. Last but not least, the EMH theory doesn't include the behavior of human being of which "demanding and supplying determine the stock prices" as well as general commodity.

Due to the above reasons as well as some others, Lux and other economist present a microscopic financial market model based on the multi-agency[3] which is the topic of the second section of this essay. Before this section, some statistical property about the real financial market such as the scaling law, fat tail distribution of return, volatility clustering are discussed in the first section; On the third section the similarity between the multi-agents based financial market(MABFM) and the Ising model (IM) in statistics mechanics is introduced. And based on the results from a modified Ising model, we will be able to reproduce the scaling law, fat tail effect in the financial markets.

2. SOME EMPIRICAL RESULTS ABOUT THE FINANCIAL MARKET

2.1. Definition of quantities

Due the well record of the financial market, people have enough data for a very accurate analysis. For example, in [2],the authors analyze 1 mininute data on the stock index, S&P 500 (the Standard and Poor index of the 500 largest stocks) for 13 years 1984-1996 period of which the total number exceeds 4.5 million. Based on the rich data resources, some empirical analysis are made to reveal some interesting statistical structure of the financial markets.

To begin with the empirical analysis, we need some definition of the quantities to characterize the system. The first kind of quantities are to describe the prices and the prices changes. One of them is logarithmic price return defined as[4]:

$$G_{\Delta t}(t) = \ln p(t + \Delta t) - \ln p(t) = \ln \frac{p(t + \Delta t)}{p(t)} \tag{1}$$

Where $p(t)$ is the stock prices (or the index) at time t and Δt is the time period varying from 1 minute to several months. As we know, stock prices changes due to inflation or other stationary factors such as currency of different country. In both cases, the all stock prices are different by multiplying a factor so called price scales for different time and different country. To make the stock prices comparable for different period and countries, we need to cancel out those factor. That is why we chose such a quantity.

Also some authors chose simple variable named price changed

$$p_{\Delta t} = p(t + \Delta t) - p(t) \tag{2}$$

The weekness is that it is influenced by the changes in price scale. So for a long time series, this variable is not a suitable good one.

Another way to represent the price changes is the price return:

$$R_{\Delta t} = p_{\Delta t}/p(t) \tag{3}$$

It is suitable variable when people want to compare the price time series of different stocks.

The second quantity are introduced to describe the correlation between return[4] (indirectly correlation between prices too) at different time which reads:

$$C(\tau) = \frac{< G(t+\tau)G(t) > - < G(t+\tau) >< G(t) >}{< G^2(t) > - < G(t) >^2} \tag{4}$$

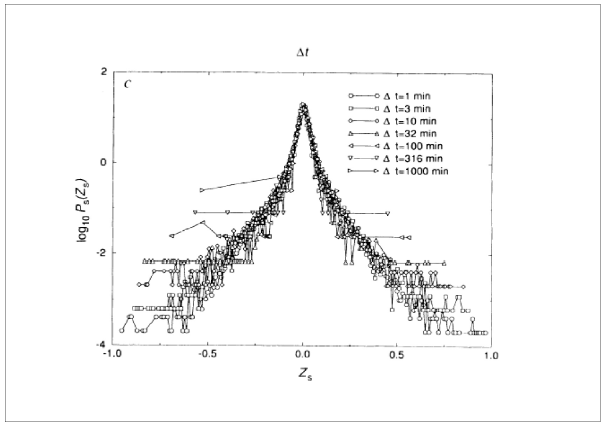

FIG. 1: Distribution probability of the S&P500 index variations

The third quantity is the volatility which describe the risk of the stock[4] defined as:

$$V_T(t) = \sum_{\tau=t}^{t+T} (G(\tau) - \bar{G}_T)^2 \tag{5}$$

where $T = n\Delta t$ and $\bar{G}_T$ is the mean value of $G_T(t)$ at the period of T, i.e.:

$$\bar{G}_T = \frac{1}{n} \sum_{t}^{t+T} G(\tau) \tag{6}$$

The forth quantity is the transaction volume $\omega(t)$ at time t.
In the following subsection, we will investigate the probability distribution of the return and the correlation.

2.2. Three main empirical results

Distribution property of return
In [5], stock index S& P500 $y(t)$ is analyzed for different time period Δt varying from 1 minute to 1000 minutes. For different Δt, index variations (just like priced change defined in the fore section) $Z_{\Delta t}(t) = y(t) - y(t - \Delta t)$ is calculated of which the frequency is counted. Finally the probability distribution of the index variation rescaled is given in the following figures(1):

As we have said the price change is not a suitable variable to describe price change for different time and different stocks. You can see the large deviation for large index variation. A similar results is shown for a more suitable variable, the logarithmic return $G_{\Delta t}(t)$ (see figure(2)). It reveals that below some certain time scale, the probability distribution is log-normal distribution(In [5],they conclude that the central region of the distribution may be well describe by a "Levy stable symmetrical distribution" which is the generalization of the normal distribution):

$$p(x) = exp[-\ln^2(x)] \tag{7}$$

And above that time scale, the distribution is power law.

$$p(x) = x^{-\alpha} \tag{8}$$

That is so called the "fat tail" in contrast to the fact that normal distribution decay very quickly after two standard deviation.

And the distribution for different time period Δt and for different stock index share the power law exponent α . That is the scaling law[5]. So we can see different index and for different country, stocks, the probability distribution of the return is controlled by a "universal class".

The distribution of firm sizes, growth rate personal income[4] and wealth[8] are also obey the log-normal center power law tail distribution with different parameters. This revokes us that there must be some important reasons or some basic laws the govern those phenomenons.

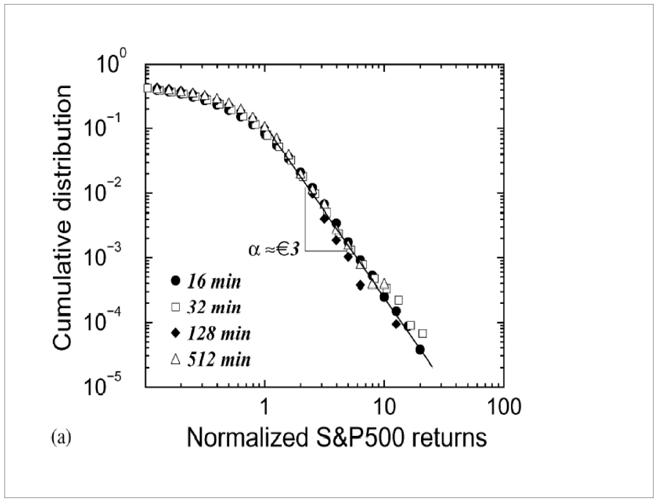

FIG. 2: The distribution function of return

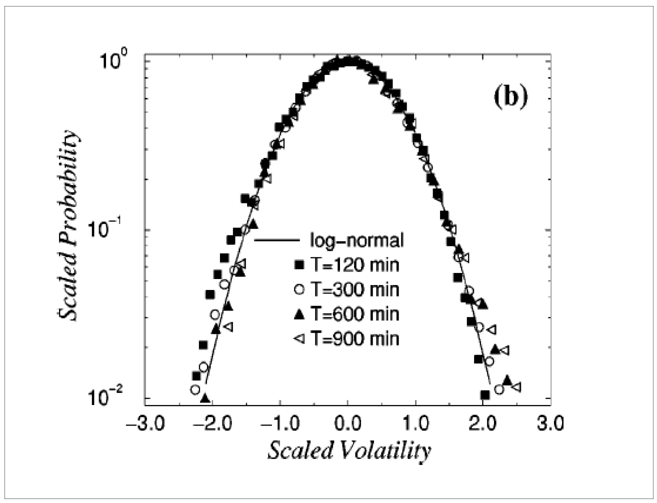

FIG. 3: Probability distribution of volatility

Volatility and autocorrelation

This part mainly consent about the volatility (or variation) and the correlation of the return. They can both be looked as the second moments of the return.

The distribution function for the volatility was found to obey the similar distribution as the distribution probability of the return[4] that is log-normal for the center part and power law for the tail[3]. That is, the probability of high volatility doesn't decay so fast a normal distribution. This means that high volatility tend to 'cluster' to prevent a fast decay. This show that they are not independent.

The autocorrelation is analyzed by spectrum method such as detrend fluctuation analysis (DFA)[4] [6], the author proved that the long-range power law correlation is the consequence of the dynamics of the economic system and not simple the results of the distribution.

transaction volume and price, demand and supply

Another important relation is between demand-supply and the prices. In general goods, we know demand and supply determine the price of the goods. For financial markets, does the same thing happen? Investigation of the book of stocks give a positive answer to this question[9]. They found that curves of the price changes $\Delta p(t)$ vs transaction volume $\omega(t)$ for different stocks can be rescaled into a master curves (see figure(**??**. This in certain sense confirm the statement that the demand-supply transaction determine the prices: if the supply is much greater than the demand the prices drop and the prices increase if there are more demand than supply. Or in a better way, the demand-supply and the prices interplay with each other.

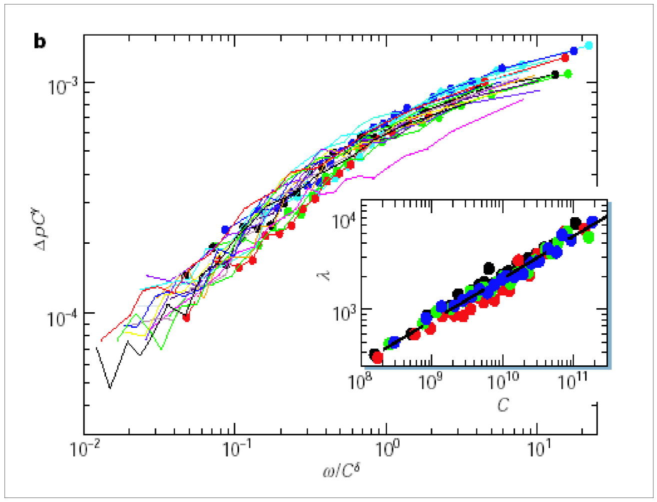

FIG. 4: The master curve for price shift vs transaction volume (rescaled with some function C)

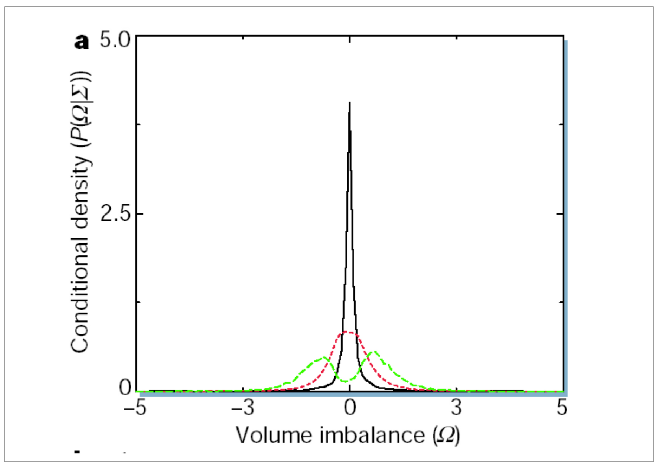

FIG. 5: The relation of the conditional probability and the local imbalance

A sophisticated study showing the different behavior of financial market for the demand and supply can be found in the paper[11]. Two parameter are defined as:

$$\Omega(t) = \sum_{i}^{N} q_i a_i \tag{9}$$

$$\Sigma(t) = < |q_i a_i - < q_i a_i > | > \tag{10}$$

Where q_i is the number of shares trade in transaction i in a short time interval Δt, $a_i = \pm 1$ denotes the buyer-initiated and sell-initiated trades respectively and $< .. >$ is the local average. The order parameter $\Psi(\Sigma)$ is defined as the peak value of the joint probability $P(\Omega|\Sigma)$(See figure (5)). From the data of 116 most actively trades stocks between 1994-1995, the author found that (See figure (6)):

$$\Psi(\Sigma) = 0 \quad for \ (\Sigma < \Sigma_c); \quad \Sigma - \Sigma_c \ for \ (\Sigma > \Sigma_c) \tag{11}$$

The above figures5,6 show that there exist two different phase of financial market. One is the equilibrium phase where demand match supply; the other is demand exceed supply or vice versa. That is the out-of equilibrium state. It is the out-of-equilibrium state of demand-supply that increase or suppress the stock prices.

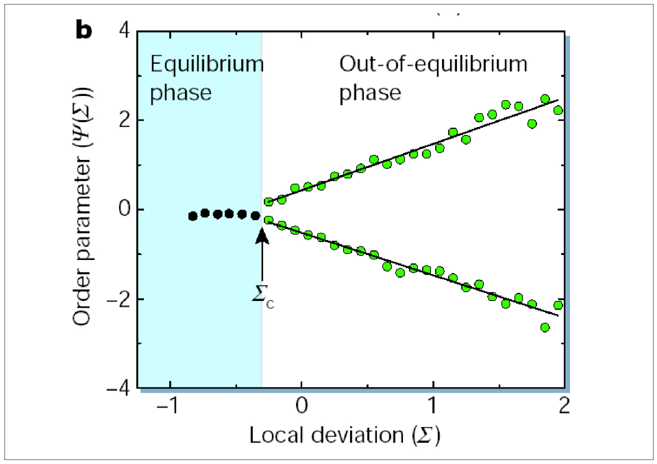

FIG. 6: Relation of order parameter and the local deviation

2.3. Analysis

From the above empirical results, we can draw several statements:

- For different time domain or frequency domain, the return or logarithmic return behaves in different ways; **(Price)**
- There exist two kind of phase for the financial markets, one is that demand match supply, the other is that one of them exceed the other. **(Demand-supply)**
- Demand and supply determine or at least affect the stock prices. **(Demand-supply and Price)**

The first statement is strange for economist is due to the central limit theorem (CLT). The CLT shows that if there are infinity number of random variables with finite mean values and finite standard deviation, the sum of those random variables has a distribution of Gaussian (or normal distribution.) This seems correct for the center part of the distribution probability of the return. The log-normal distribution is a normal distribution if we consider log function of the variable as a new variable. That is the logarithmic return $\ln p(t)$. So we can see for small time period logarithmic returns are independent random variables for different stocks. This result is consistence with CLT theorem. And we can also see that not stock price but logarithmic return is a random walk process for small time period or high frequency.

But for the power law tail, the results are not consistence with the CLT. This could be interpreted by the assumption that the logarithmic return for different stocks are not independent, that is to say, the logarithmic return for different stocks are strong correlated. This judgement can also be made from the volatility distribution. For a normal distribution, the distribution decay rather quickly after two standard deviation. But for a strong correlated stock prices, they behave in a similar way (the positive correlation are shown in many cases). This make the probability high increase, that is the origin of fat tail.

According to the third statement, we know the stocks prices are strong correlated due to the strong out-of equilibrium of the demand and supply. That is to say people tend to buy or sell the stocks in the similar way in some situation. That is the herding behavior in the financial market. This can also be revealed in the second statement that there exist out-of equilibrium state in the financial market.

In short, we have a frame in our mind that:

- **Agent dynamics :** The people's behavior in financial market can be divided into two case, equilibrium state or out-of-equilibrium state, ie. supply match or highly exceed demand or vice versa. The later behavior is herding;(How do people buy and sell, How does herding appear?)
- **Agent-price mechanism :** The in or out-of equilibrium behavior of the demand-supply interplay or highly affect the stocks prices;(How does demand-supply or buy-sell determined the stock prices?)
- **Price dynamics :** The equilibrium of demand and supply give rise a log-normal distribution of return for small time period, i.e. center part of the distribution. The highly out-of-equilibrium of demand and supply will give rise

a strongly correlated behavior of the return of stocks in large time scale. This is the power law fat tail.(How do price change? How to interpreted the empirical results?)

So if we want to construct any theory, we need to consider the above three elements. In the following, we use our logic to understand the multi-agents based financial market model.

3. MULTI-AGENT BASED FINANCIAL MARKET

3.1. The model

There are many version of multi-agency based financial market (MABFM) model. The one introduced here is presented by Thomas Lux[3]. There are three basic elements:

The first element: Agent dynamics

The agent here are the traders who buy or sell stocks according their strategies. There are three kind of strategies.

- **fundamentalist** : This kind of trader trade according to the fundamental value of the stocks. If the prices is high than the fundamental values, they sell; if the prices is lower than the fundamental values, they buy stocks. They believe the fundamental value describe the true values of the stocks.
- **optimism** : This kind of trader trade according the trend of the price of the fore time step. The the price increases at fore step, he buy; if the price decrease he sell. They are optimism and believe that the trend will continue.
- **pessimism** : This kind of trader trade according the trend of prices too, but if the price increase, they sell instead of buy for the optimism. and sell when the price decrease. Both the optimism and pessimism are all the so called 'noise trader' which are doing the technique analysis.

In a simple case the author consider a market consisted of only optimism with number of n_+ and pessimism with number of n_- due to larking knowledge of fundamental value. The total number of trader is $2N$ and $n = (n_+ - n_-)/2$, $x = n/N$. Every trader hold the same number of stocks and they can only buy or sell all the stocks they own. Another assumption is that the trader will change their attitude according some rules. The optimism change to a pessimism in a probability of p_{-+} and p_{+-} is the probability of opposite transition.They are assumed to depend only on x.

A reasonable assumption for p_{+-} and p_{-+} is that the more people, the higher the probability to change. So we have:

$$\frac{dp_{+-}}{p_{+-}} = adx \quad ; \quad \frac{dp_{-+}}{p_{-+}} = -adx \tag{12}$$

Where a is a constant and minus sign is add to conserve the total number. Solving the equation, we have:

$$p_{-+}(x) = \nu e^{ax} \quad ; \quad p_{-+}(x) = \nu e^{-ax} \tag{13}$$

From definition of n and x, we have the following equation:

$$\frac{\partial x}{\partial t} = [(N-n)p_{+-}(x) - (N+n)p_{-+}(x)]/N$$

$$= (1-x)p_{+-}(x) - (1+x)p_{-+}(x) = (1-x)\nu e^{ax} - (1+x)\nu e^{-ax} \tag{14}$$

$$= 2\nu[\sinh(ax) - x\cosh(ax)] = 2\nu[\tanh(ax) - x]\cosh(ax) \tag{15}$$

Investigation of this equation of motion, we will be able to get the fixed point for this equation.

- For $a \leq 1$ equation has a uniques stable equilibrium point at $x = 0$ which means the buyer and seller has the same number. That is a equilibrium state of demand and supply.
- For $a > 1$ we have three fixed point, $x = 0$ is a unstable one and two additional stable fixed point is x_+,x_- with $x_+ = -x_-$. These two stable fixed point represent two out-of equilibrium state of demand and supply. That is demand highly exceed supply or vice versa. That is the herding behavior in financial market.

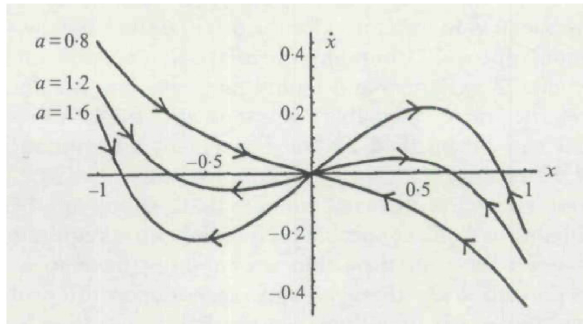

FIG. 7: The typical agent dynamics curves

Three typical dynamics curves are plotted in the figure(7):

These two fixed point can be calculated from the equation:

$$\tanh(ax) = x \tag{16}$$

If initially, the financial market is at $x = 0$, a small deviation will destroy this equilibrium state and cause a "snowball-like cumulative process" then decay to a stable bullish or bearish financial market.

In fact, from above analysis and the equation(16), we can clearly see that this model is just a replica of our Ising model in statistical mechanics where $S_i = 1$ represent a optimism and S_i represent a pessimism. We will discuss it in the later section.

The second and third elements: Market-maker and price dynamics

In the above, we only discuss the behavior of agent or traders. We haven't consider the effect of demand-supply to the price which will lead to the price dynamics.

When the demand matches the supply, the stocks prices won't change and the transaction is done. But if the supply and the demand doesn't match each other, how can the transaction be done? In real life, we need the exchange center to connect both side to make a deal. Here, the author consider a new elements of financial market

- **Market-maker**: This kind of people connect both side of demand and supply and change the prices to according to a certain rules to match the demand and the supply and make the transaction succeed.

For simplicity, we assume that all the traders can only buy or sell a fixed number of stocks t_N. Then the net excess demand can be written as:

$$D_N = (n_+ - n_-)t_N = 2nt_N = 2Nxt_N = xT_N \tag{17}$$

Where T_N is defined as $2Nt_N$.

Because of the excess of demand or supply, we cannot make the transaction done. Then we have to introduce one more group. And remember that we still have the fundamentalist group that doesn't appear in the financial market. So we chose the fundamentalist to balance the demand and supply. The excess demand of this group depends on the difference between the fundamental values p_f and the market values p. The simplest form of excess of demand could be assumed to:

$$D_F = T_F(p_f - p) \tag{18}$$

Then the market-maker will adjust the stock prices to match the demand and supply. Thus the price dynamics equation is:

$$\frac{\partial p}{\partial t} = \beta(D_N + D_F) = \beta[xT_N + T_F(p_f - p)] \tag{19}$$

Where β is the speed of adjustment coefficient. Market clearing would result in a equilibrium price:

$$p^* = (T_N/T_F)x + p_f \tag{20}$$

A more careful consideration will make us to change the transition probability to a more suitable way, that is, it should depend on the price changing rate:

$$p_{-+}(x) = \nu e^{a_1\dot{p}/\nu + ax} \quad ; \quad p_{-+}(x) = \nu e^{-a_1\dot{p}/\nu - ax} \tag{21}$$

Now the system is completed with both agent dynamics and price dynamics. The equation reads:

$$\frac{\partial x}{\partial t} = 2\nu[(a_1\dot{p}/\nu + a_2x) - x]\cosh(a_1\dot{p}/\nu + a_2x) \tag{22}$$

$$\frac{\partial p}{\partial t} = \beta(D_N + D_F) = \beta[xT_N + T_F(p_f - p)] \tag{23}$$

For the evolution of this system, we have the following results using $E = (x, p)$ to represent the system state.

- If $a_2 \leq 1$ and $2[a_1\beta T_N + \nu(a_2 - 1)] - \beta T_F < 0$: $E_0 = (0, p_f)$ is the only stable fixed point;
- If If $a_2 \leq 1$ and $2[a_1\beta T_N + \nu(a_2 - 1)] - \beta T_F \geq 0$: $E_0 = (0, p_f)$ is the unstable fixed point, but at least one stable limit cycle exists;
- if $a_2 > 1$: E_0 is a unstable fixed point, and two additional stable fixed point emerge, $E_+ = (x_+, p_+)$ and $E_- = (x_-, p_-)$ with $x_+ = -x_-$ and $p_f - p_- = p_+ - p_f$.

3.2. The numerical simulation of MABFM

Beside his theoretical model, Lux and his collator also make numerical simulation on this model which appears on Nature[10]. Some interesting results are shown in this actificial financial market:

- log-normal center and power law fat tail distribution of the distribution probability of return.
- The volatility clustering.

Their results agree well with the empirical results.

4. ISING MODEL AS A TOY MODEL OF MABFM

4.1. Similarity and difference of IM and MABFM

From the above, we can see some similar structure of the Ising model and MABFM. The followings are something similar between two model:

- $S_i = \pm 1$ in IM can be mapped into Buy or Sell (optimism or pessimism) in MABFM;
- $M = \sum_i^N S_i$ can be mapped into $D_N = (n_+ - n_-)t_N$ in MABFM. M can work as a order parameter to seperate two phase: Paramagnetism (PM) and Ferromagnetism (FM). In MABFM these two phase correspond to equilibrium market or out-of-equilibrium market and the order parameter is Σ in the second section[11].

We also need to clarify the difference between these two models:

- In IM, spin interact with nearest neighbor spin. And in MABFM, the concept of space is unclear. So perhaps we need a infinity range interaction IM. A IM with finite interaction range can also be viewed as a model for trader who share certain information in a small group.
- In IM, we only consider two state, up or down corresponding buy or sell, also, optimism or pessimism. We cannot find a correspondence in IM to fundamentalist.
- In IM, we cannot find a correspondence of the stock price.

4.2. A modified Ising model

Despite the above defect of IM when we map IM to MABFM. As we have discussed in section one, a successful model of MABFM must have three elements, the agent who buy and sell and the market-marker who match the demand and supply and consequently, the price dynamics .

The first elements: Agent dynamics

In Ising model,we will be able to using the mapping between this two model to simulate the dynamics of agent. Bearing in our mind that the $S_i = \pm$ mapping to buying or selling, we can write down the Hamiltonia for this MABFM system according the following two reasons:

- A agent will do what a group around him do. That is the case that a small group of people share the same information and use the same strategy to trade. This will give rise a term $\sum_{i,j} J_{i,j} S_i S_j$.
- In a financial market, for a short period the net true value of the stocks doesn't change which means a financial market is a zero sum game theory. Someone win, someone else must lost. So to have more change to be a winner, the trader have to be in the minority of the financial market, that is when most of the people buy, you have more chance to win if you sell. That is the so called minority game. This will results in a term like $-\alpha C(t) \frac{1}{N} \sum_{i=1}^{N} S_i$. where $C(t)$ represent a strategy exponent.

Based on the above reasoning a modified Ising model presented by Bornholdt.[14].

$$H = \sum_{i,j} J_{i,j} S_i S_j - \sum_{i=1}^{N} S_i \alpha C(t) \frac{1}{N} \sum_{j=1}^{N} S_j \tag{24}$$

With the help of quantum mechanics and statistical mechanics,we will be able to get evolution of $S_i(t)$. We can also modified the Ising model to any sentence if we want, perhaps the spin glass.

The second and third elements: price dynamics

The left thing is how to connect the agent dynamics to the price dynamics. Here, we have to introduce some relation used in the finance[3, 15] which appears also in second section. The excess demand is balanced by the fundamentalist and the market-marker will adjust the price to a market clearing values:

$$T_F[\ln p_f(t) - \ln p(t)] + T_N M(t) = 0 \tag{25}$$

Which is little difference from the equation(19) by replacing p by $\ln(p)$. We can calculate clearing values from above equation:

$$\ln p(t) = \ln p_f(t) + T_N/T_F M(t) \tag{26}$$

So the return can be got if we know the fundamental values $p_f(t)$ and $M(t)$ which reads:

$$r(t) = \ln p(t) - \ln p(t - \Delta t) = [\ln p_f(t) - \ln p_f(t - \Delta t)] + T_N/T_F[M(t) - M(t - \Delta t)] \tag{27}$$

Here we can also see the some property of some econophysics theory, they are made of a half physics (agent dynamics in this case) and a half economic(market-marker and price dynamics in this case).

Finally, if we know all the $r(t)$ values we can do the distribution probability analysis. Many modified Ising model calculated using the above method really show the statistical property of the financial markets.

5. CONCLUSION AND ACKNOWLEDGE

In the essay, the statistical property of the real financial market is introduced on the base of several empirical investigations. These statistical property includes distribution with log-normal center and power law tail, scaling, volatility and long-range auto-correlation. The relation between these property is analyzed which can be divided into three part: the herding behavior cause the out-of equilibrium between the demand and supply; demand and supply determined the stock price; so the interacting agent behavior results in a power law fat tail.

Based on the above analysis, we introduce the multi-agent based financial market model. The main elements of such model is the agent dynamics and the price dynamics. The model presented by Lux is given in detail which reveal some of the statistical property of real financial market.

The similarity between Ising model and the multi-agent based financial market model results in a map between the Ising model and the agent dynamics in MABFM. With the help of statistics mechanics and quantum mechanics we can solve some the modified Ising model and reveal some similar statistical property of real financial market.

Physicist is the most aggressive scientist in all the area of science, they always try to understand the whole world in their own way. The econophysics is a typical example of their ambition. This essay is a example for me to understand the econophysics way.

This essay is a combination of several paper. But I understand them in my own way and in my own logic. Some opinion are not from any paper listed and may be incorrect.

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