Critical Phenomena in Percolation Theory

Patricio Jeraldo

December 13, 2005

Abstract

In this paper, the critical behavior of the percolation models is reviewed, with emphasis on the scaling laws present and its relationship with other simple models like the Ising magnet. A liitle bit of the history and some examples of its application are given to illustrate the range of phenomena that can be described with such models.

1 Introduction

Percolation theory refers to a class of models that describe the properties of a system given the networking among its constituents. There are two basic types of percolation models. In the first one, points are defined on an underlying lattice in such a way that, in every lattice site there is a probability p for a point to exist there. This is the *site percolation* model. In the second one, bonds are defined between two neighboring sites on a lattice. Each bond has a probability p to exist. Accordingly, this model is called *bond percolation*. In both cases, structures of connected points can be defined (clusters), in a way that it is possible to create a path between any two points of the cluster. As the probability p is increased, larger clusters will be formed. Eventually, a cluster that has a path that spans the whole system will be formed. If the lattice is infinite in extension, the size of this particular cluster would be infinite. The value of p that creates this cluster is called the *critical probability*, denoted by p_c. We can think of this model as a porous wall that separates two volumes, one of which is a fluid, and the bonds between sites can be thought of as the pores inside the wall. Then the probability p would correspond to the porosity of the wall. The critical porosity then is the smallest value of the porosity such that fluid percolates to the other side. Hence the name *percolation* for this model.

In the case of periodic or infinite lattices, the existence of a diverging quantity, namely the (average) cluster size, at a specific, finite value of a parameter, and the qualitative change in the behavior of the system after crossing this value tell us that one should look for critical behavior. In fact, near p_c, several quantities exhibit power-law behavior, and there are scaling laws relating the different critical exponents. The whole machinery used to study phase transitions and critical phenomena can be used to understand how percolation works. Since this is a probabilistic system, the relevant parameter akin to temperature is the probability p.

The first of such models was introduced in the 1940s (Flory 1941, Stockmayer 1943), and it was used as way to explain polymerization phenomena that leads to *gelation*, that is, the existence of a network of chemical bonds spanning the whole system. Later, in the late 1950s (Broadbent and Hammersley 1957) the term “percolation” was coined and associated to this particular set of geometrical models. Its applications range from its rather obvious use as a model for fluids in porous media to the study of the effect of dopants in semiconductors, and also as a way to understand the propagation of forest fires. The relationship of percolation to critical phenomena was only emphasized since the 1970s (Essam and Gwilym

1971).

2 The one-dimensional percolation problem

Although limited in scope, the one-dimensional percolation problem is useful to get an idea of the main features of this system.

Let there be a one-dimensional, infinitely long chain with sites defined in fixed distances on the chain. Each site is occupied randomly, with probability p. A cluster is defined as a set of neighboring occupied sites with no empty sites in it. Two neighboring clusters are separated by at least one empty site. Since a cluster requires the existence of two empty sites, one at each side of the cluster (each with probabily $(1-p)$), the number of clusters of size s per lattice site is

$$n_s = p^s(1-p)^2$$

Since this is also the probability for a lattice site to be the left end of a cluster of size s, we see that the probability for a site to be part of a cluster of size s is $n_s s$.

Now we can deduce the percolation treshold. We are looking for a cluster spanning the whole system. For any $p < 1$ there is at least one empty site, and therefore there is no continuous chain of occupied sites spanning the system. On the other hand, for $p = 1$, all the sites are occupied. Thus, for the one-dimensional system,

$$p_c = 1$$

A consequence of this is that there is no way to study the system for $p > 1$. Nevertheless, just like in the one-dimensional Ising magnet, we can extract more information from the behavior near $p = p_c$. Of particular interest is the *mean cluster size*, defined as

$$S[p] = \sum_s \frac{n_s s^2}{\sum_s n_s s}$$

where the sum runs from $s = 1$ to infinity. In this case,

$$S(p) = \frac{1+p}{1-p}$$

As expected, the mean cluster size diverges as $p \to p_c = 1$. Another interesting quantity is the *pair connectivity* (or pair correlation function) $g(r)$, defined as the probability that a site at a distance r belongs to the same cluster. In this case,

$$g(r) = p^r$$

We can also write

$$g(r) = \exp\left(-\frac{r}{\xi}\right),$$

where

$$\xi = \frac{1}{\ln(p)} \simeq \frac{1}{p_c - p}$$

The power law behavior of ξ is only valid near $p = 1$. This form for ξ is familiar in the sense that the correlation length diverges as a power law near a phase transition. We can also see that the relevant scaling parameter is the probability. When comparing to thermal phase transitions, we see that the mean cluster size S is analogous to the susceptibility of the system.

3 Percolation models in more than one dimension

In lattices with more than one dimension, the percolation treshold exists for $p_c < 1$. In fact, exact calculations of p_c exist for some two-dimensional lattices. In particular, for the Bethe lattice (or Cayley tree)[1] the critical probability p_c depends on the z value of the nearest neighbors as

$$p_c = \frac{1}{z - 1}$$

The values of the critical exponents obtained for the Bethe lattice are the limiting case when the dimensionality of the system goes to infinity.

As we will see, the scaling hypothesis seems to hold for the percolation model. Given the results for one-dimension and for the Bethe lattice, it is possible to state that, in general,

$$n_s \propto s^{-\tau} \exp(-cs),$$

where τ is a free exponent, and c is a function of p. Near the percolation treshold, c is allowed to behave as a general power law

$$c \propto |p - p_c|^{1/\sigma},$$

where σ is another free exponent. Defining P, the percolation probability, as the probability of a site to belong to the infinite cluster, it can be shown that, near the percolatio treshold

$$P \propto c^{\tau-2} \propto (p - p_c)^{(\tau-2)/\sigma} = (p - p_c)^{\beta}.$$

[1] In the Bethe lattice, each site is connected to z nearest neighbors in a way that no closed loops are possible.

Hence, β is defined as

$$\beta = \frac{\tau - 2}{\sigma}.$$

Similarly, for the mean cluster size, it can be shown that it diverges as

$$S \propto c^{\tau-3} \propto |p - p_c|^{(\tau-3)/\sigma} = |p - p_c|^{-\gamma},$$

and the exponent γ is defined as

$$\gamma = \frac{3 - \tau}{\sigma}.$$

Finally, the kth *moment* of the cluster size distribution is defined as

$$M_k = \sum_s s^k n_s.$$

Near the percolation treshold, M_k also behaves as a power law,

$$M_k \propto c^{\tau-1-k} \propto |p - p_c|^{(\tau-1-k)/\sigma}.$$

In particular, for the case $k = 0$, that is, the mean total number of clusters,

$$M_0 \propto c^{\tau-1} \propto |p - p_c|^{(\tau-1)/\sigma} = |p - p_c|^{2-\alpha},$$

so we have the following relation among $\alpha, \beta, \gamma, \sigma$ and τ:

$$2 - \alpha = \frac{\tau - 1}{\sigma} = 2\beta + \gamma.$$

This is the familiar Rushbrooke scaling law.

After all this, we can summarize the correspondence between the quantities characterizing percolation theory and the Ising magnet. The mean number of clusters is analogous to the zero-field free energy, the percolation propability behaves as the spontanous magnetization, the mean size of clusters plays the role of the susceptibility and the pair connectivity is analogous to the pair correlation function.

One last comment about the exponents. It seems that the critical exponents and some other variables are not functions of the lattice but only functions of the dimensionality of the system, and also it seems that the values of the exponent are shared by both site and bond percolation models. This reveals that the percolation model is a universality class on its own. Also, the value of the critical exponents seem to converge to the value of the ones obtained for the Bethe lattice as the dimensionality of the system goes to infinity, hence validating its role as a model for the limit $d \to \infty$.

4 Applications, experiments

4.1 Droplet description of critical phenomena

One of the applications of percolation theory is in the quiantitative analysis of the droplet description of critical phenomena[5]. This description asserts that, when two phases are in equilibrium, droplets of one phase nucleate inside the other phase. As the droplets form, they tend to grow. Eventually, the system goes to the new phase. Although the droplet model is very old, the successful application of percolation theory (in three dimensions) only came tin the late 1980s (the Swendsen-Wang description).

In the case of the zero-field diluted[1] Ising model for the limit $T \to 0$, the percolation model applies, with the caveat that an appropiate definition for the clusters should be used: for two spins to belong to the same droplet, they have to be connected with an additional bond. These bonds are distributed randomly in the lattice, with a temperature dependent probability $\pi = 1 - \exp(-2J/kT)$. These bonds are not used in the calculation of the interaction energy. Results from simulation show that the droplets have fractal dimension $D = 2.5$. Theory predicts that $D = d - \beta/\nu$ with the Ising values for the critical exponents, yielding $D = 2.53$. The results seem to agree, despite a technical issue beyond the scope of this paper.

4.2 Swendsen-Wang algorithm

The Swendsen-Wang algorithm is very useful for simulating Ising-like systems near the critical point. Instead of flipping single spins, the algorithm flips entire clusters with $p = 1/2$. Then, the system has a very short relaxation time compared to the ones obtained with the Metropolis algorithm. This in results in a reduction in calculation time needed to compute reliable averages.

4.3 Mixture of conducting and insulating spheres

This experiment[3] consists of a large number of plastic, insulating spheres mixed with a fraction p of spheres coated with metal. The conductivity of the system is then measured as a function of p and also as a function of the applied pressure to the contact. This is needed to take into account the effect of the random packing of

[1] The diluted Ising model is a model with interacting spins on a lattice plus a fraction p of non-magnetic sites

spheres, and its effect on the contact between spheres. The authors want to check if a percolation model is a valid description of this system.

The behavior of the percolation probability $P(p)$ near the percolation treshold is

$$P(p) \sim |p - p_c|^\beta,$$

with $0 < \beta < 1$ and, experimentally, the conductivity behaves as

$$\sigma(p) \propto (p - p_c)^\mu$$

with $\mu \sim> 11$. Thus near the treshold, $\sigma(p)$ grows slower than $P(p)$. This is because the "arms" of the infinite cluster do not contribute in a significant way to the conductivity of the system.

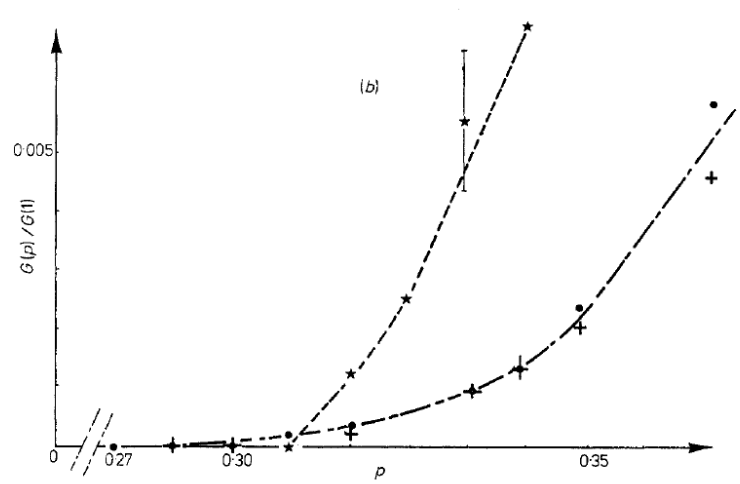

Figure 1: Measured conductance measured as a function of p, normalized to the value obtained at $p = 1$. Notice how it vanishes as p gets closer to 0.30.

The authors find that the measurements are consistent with $p_c \simeq 0.30$ (see fig. 1) and that the conductance is also a function of the applied pressure (without changing the percolation treshold). The latter was an indication that bond effects are important, despite that is easy to think of this system as a site percolation system. A better way to model this system is to consider a mixed, site-bond percolation model.

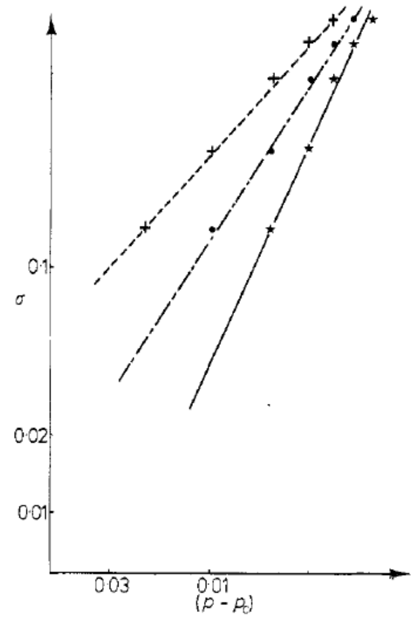

Figure 2: Behavior of conductivity near treshold. Due to uncertainties, the calculated value for μ varies from 0.9 to 1.25.

4.4 Forest fire propagation

For this model[6], the authors propose a percolation-like model with weighted sites. The use of weights is justified since, for instance, a burnt tree is less likely to burn a neighboring tree than a burning tree. The authors study the rate at which the fire spreads and the coverage of the fire using numerical simulations.

The authors find, among other things, that the percolation treshold depends on the propagation coefficient as a power law (see fig. 3). The authors also find that, for the case when a fraction p of trees are ignited initially, there is a dependence of the fractal dimension of the area covered by the fire on this initial fraction p (see fig. 4). In fact, when p reaches the percolation treshold, the fractal dimesion reaches 2, hence the fire covers all the field. This can be used as a criteria to determine p_c in systems described by similar models.

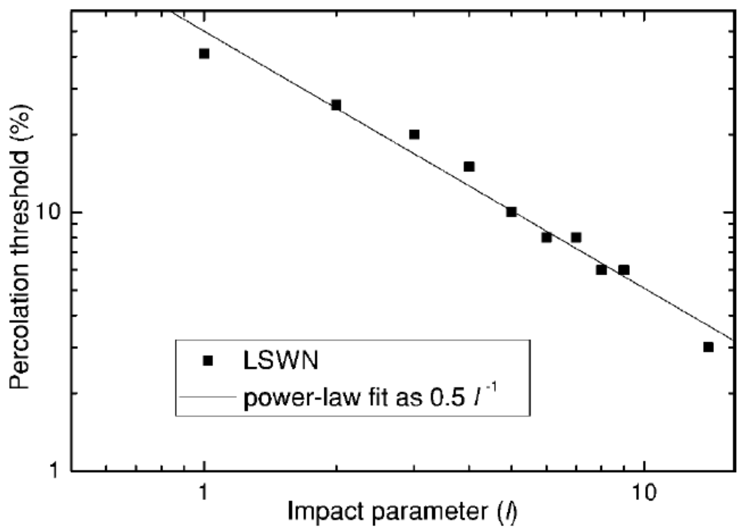

Figure 3: Percolation treshold as a function of impact parameter (propagation coefficient).

5 Conclusion

The percolation model was explained. It was shown that it obeys the same scaling laws as any other critical phenomena and the usual tools used for analysis should work, but the relevant quantity is the site or bond probability instead of the temperature. Finally, some examples of its use were given as a way to show the broad range of applications. Given the number of recent papers using or extending the model to other geometries or research areas, it is fair to say that percolation theory is still an active topic, and it would be interesting to follow its developement closely.

6 Addendum

This section includes a couple of figures that, because of technical reasons, could not be included in the first part of the paper.

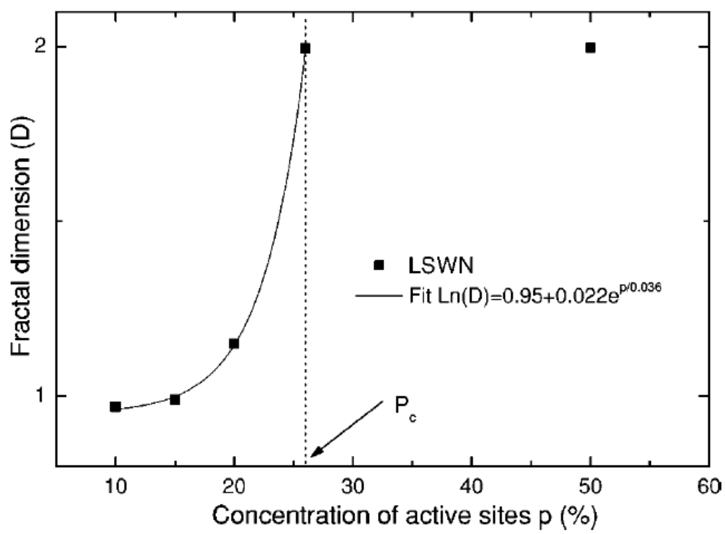

Figure 4: Fractal dimension of fire-covered area as a function of the fraction of initial burning sites.

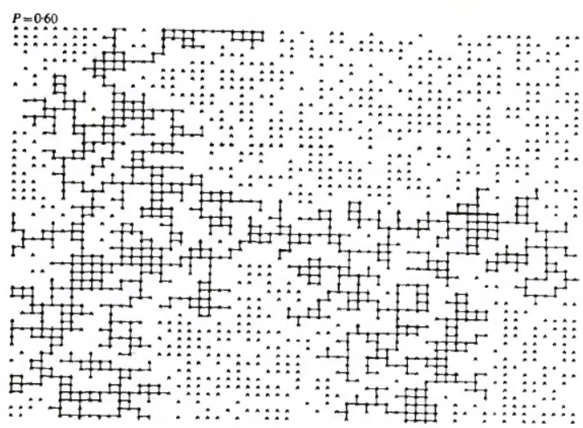

Figure 5: A site percolation model with p = .60, just above the percolation treshold. The largest cluster is shown. Note how each wall has at least one point in the (percolating) cluster

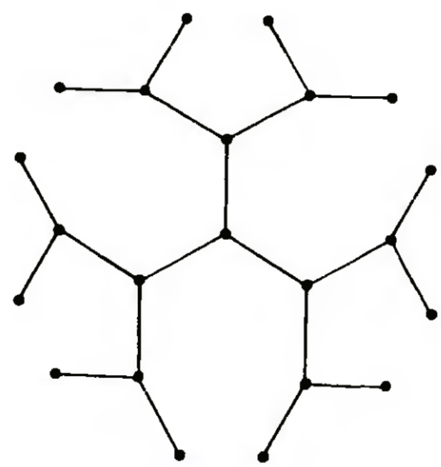

Figure 6: An example of a small Bethe lattice, with $z = 3$.

References

[1] JW Essam. Percolation theory. *Rep. Prog. Phys.*, 43:53, 1980.

[2] S. Kirkpatrick. Percolation and conduction. *Rev. Mod. Phys.*, 45:53, 1973.

[3] H. Ottavi, J. Clerc, G Giraud, J. Roussenq, E. Guyon, and C. Mitescu. Electrical conductivity in a mixture of conducting and insulating spheres: an application of some percolation concepts. *J.Phys. C: Solid State Phys.*, 11:1311, 1978.

[4] M. Sahimi. Flow phenomena in rocks: from continuum models to fractals,percolation, cellular automata, and simulated annealing. *Rev. Mod. Phys.*, 65:1393, 1993.

[5] D. Stauffer and A. Aharony. *Introduction to Percolation Theory*. Taylor and Francis, 2nd edition, 1994.

[6] N. Zekri, B. Porterie, J-P. Clerc, and J-C. Loraud. Propagation in a two-dimensional weighted local small-world network. *Phys. Rev. E*, 71:046121, 2005.