Physics 504: Statistical Mechanics and Kinetic Theory
HOMEWORK SHEET 8

Due 6pm Mon May 4 2020

Question 8–1.
Consider a fluid system described by the grand partition function

$$\Xi(\lambda, V) = (1+\lambda)^V \left(1+\lambda^{\alpha V}\right),$$

where $\alpha > 0$ and V is the volume of the system in some units, such as the hard core radius, and λ is the fugacity. This form of Ξ is chosen to illustrate some points of principle; I do not know if there is a real system with this form of Ξ.

(a) Write down the equation of state in parametric form (i.e. in terms of λ), and sketch the graphs of p and v^{-1} versus λ, in the thermodynamic limit. Show that there is a first order transition, and find the specific volumes of the two phases at coexistence.

(b) Find the roots of $\Xi(\lambda, V) = 0$ in the complex λ plane at fixed V. Show that they converge to the point on the real λ axis at which the phase transition in (a) occurred, in the thermodynamic limit.

(c) Find the equation of state in the gas phase, and notice that it exhibits no signature of the phase transition as the volume per particle is decreased beyond that where the phase transition occurs according to (a). Explain.

(d) Check that the form of the grand partition function, which we have assumed, is physically acceptable, by calculating the isothermal compressibility, and explain the significance of this calculation.

Question 8–2.
This question concerns the mean field theory for the nearest neighbour Ising model on a d-dimensional hypercubic lattice with coordination number $z = 2d$, where d is the spatial dimensionality.

(a) Let M be the magnetization per spin. By writing each spin variable $\sigma_i = M + (\sigma_i - M)$ and ignoring terms in the Hamiltonian which are quadratic or higher order in the fluctuations, show that the Helmholtz free energy per spin $f(H, M, T) = JzM^2/2 - k_B T \ln[2\cosh((JzM + H)/k_B T)$. For $H = 0$ and $M \ll 1$ expand the free energy/spin up to and including $O(M^4)$. What happens to the quadratic term at T_c? Plot the functional form of $f(0, M, T)$ for the separate cases of $T > T_c$ and $T < T_c$.

(b) Hence explain why the solution $M = 0$ is not physically acceptable in the mean field theory of the zero field Ising ferromagnet below T_c. How is this reflected in the behavior of the isothermal susceptibility as calculated about the $M = 0$ state above and below T_c?

(c) Calculate the isothermal susceptibility χ_T for zero external field H in the critical region for $T < T_c$ and verify that the way it diverges very close to T_c is of the form

$$\chi_T \sim \frac{A_-}{T_c - T}, \quad \text{as } T \to T_c^-$$

and determine A_-.