Physics 504: Statistical Mechanics and Kinetic Theory
HOMEWORK SHEET 6

Due 5pm Tue 14th April 2020 online.

In this homework, you are asked to sketch certain functions. Please note that sketch means that you do NOT plot using Mathematica or Matlab, but that you use your analysis skills to figure out the asymptotics of the functions, limiting forms, zeros, etc. In other words, estimate what the function looks like without plotting it.

Question 6–1.

Consider a quantum paramagnet of N magnetic moments with $L = 0$ and $S = 1/2$.

(a) Calculate the contribution to the entropy from the magnetic moments, and sketch the result as a function of $1/x$, where $x = g\mu_B JH/k_BT$. By sketch, I mean figure out what the graph looks like, **not** plotting it on the computer!

(b) Now suppose that this system is at temperature T_1 and in a field H_1, and is thermally isolated from the surroundings. What happens to the temperature as the external field is reduced to zero?

(c) This phenomenon is a powerful experimental technique. What practical limitation does it have? Briefly suggest how the method may be improved.

Question 6–2.

Photons are quantum mechanical bosons with unit spin, 2 helicity states (the quantum analogue of polarisation), whose number is not conserved. In this question, we will study the statistical mechanics of a gas of such particles, and derive the black body radiation spectrum from the "particle" point of view, as opposed to the "wave" point of view presented in lectures.

(a) What is the chemical potential of a gas of photons?

(b) Write down the partition function for a gas of photons with states labelled by momenta $\mathbf{k}$ with energies $E_{\mathbf{k}}$ and occupation numbers $n_{\mathbf{k}}$.

(c) For photons, the dispersion relation is $E_{\mathbf{k}} = \hbar c|\mathbf{k}|$. Hence write down the free energy of a gas of photons in a volume V. You may use the fact that $\int_0^\infty x^2 \log\left(1 - e^{-x}\right)\, dx = -(2\pi)^4/720$.

(d) Hence calculate the average energy of the gas. You should recover the result obtained in lectures by another method.

Question 6–3.

Surface waves on liquid helium, at angular frequency ω and wavenumber $k = 2\pi/\lambda$ satisfy the following relation: $\omega^2 = [\sigma/\rho]k^3$ where σ is the surface tension of liquid helium and ρ is the density of liquid helium. Suppose that the liquid helium is at temperature T, and has surface area A. Show that the surface energy $U(T)$ — *i.e.* the energy of the surface waves — is given by

$$U(T) = CI\left(\frac{k_BT}{2\pi\hbar}\right)^\alpha$$

where $I = \int_0^\infty dx x^{4/3}/(e^x - 1) \approx 1.68$, and determine the coefficient C and the value of the exponent α.

Question 6–4.

In this question, you will calculate the density matrix for a free particle of mass m in a box of volume $V = L^3$, with periodic boundary conditions. This question has a succession of little steps, to help you fully understand quantum mechanics formalism.

(a) Write down the energy E eigenfunctions $\phi_E(\mathbf{r})$ in the position space representation, taking care to specify the allowed values of the quantum numbers.

(b) Write down the formal expression for the density operator $\hat{\rho}$ of the canonical ensemble in terms of the bras and kets $|E\rangle$ which are eigenkets (bras) of the Hamiltonian.

(c) Write down the matrix element $\rho_{\mathbf{r}\mathbf{r}'}$ of $\hat{\rho}$ between $|\mathbf{r}\rangle$ and $|\mathbf{r}'\rangle$, which are quantum states in which the particle is localised at positions $\mathbf{r}$ and $\mathbf{r}'$ respectively. Your answer should be in terms of $\phi_E(\mathbf{r})$.

(d) Hence show that

$$\rho_{\mathbf{r}\mathbf{r}'} = \frac{1}{V} \exp\left[-\frac{\pi\left(\mathbf{r}-\mathbf{r}'\right)^2}{\Lambda_T^2}\right]$$

where Λ_T is the thermal de Broglie wavelength $h/\sqrt{2\pi m k_B T}$. What is the partition function Z?

(e) What is the probability for the particle to be at a given position $\mathbf{r}$ in the box? Is the answer in agreement with your expectations?

(f) Using the density matrix, calculate the average energy of the particle. Is the answer in agreement with your expectations?

(g) Let $\hat{\sigma} \equiv \exp -\beta\hat{H}$ where $\beta = 1/k_B T$ as usual. By working in the energy reresentation or otherwise, show that $\hat{\sigma}$ obeys the "Schrödinger" equation, but with a time $t = i\beta/\hbar$:

$$-\frac{\partial\hat{\sigma}}{\partial\beta} = \hat{H}\hat{\sigma}.$$

This is a first order equation in β. What is the "initial condition" on $\hat{\sigma}$?

(h) Write down the Schrödinger equation above in the position space representation, *i.e.*, what is the equation for the matrix element $\sigma_{\mathbf{r}\mathbf{r}'}$? State the initial condition.

(i) Solve the above equation and compare with your answer to (d); you should of course find that $\sigma_{\mathbf{r}\mathbf{r}'} = Z\rho_{\mathbf{r}\mathbf{r}'}$.

Question 6–5.

This question concerns the virial theorem, which we proved for a system in thermal equilibrium. Here, we are concerned with the virial theorem for an isolated classical system that is not necessarily in thermal equilibrium: we shall prove it by considering the *dynamics* of the system. Thus, thermal equilibrium is a sufficient but not a necessary condition for the theorem to apply. We define the virial $\mathcal{V} \equiv \sum_i \mathbf{f}_i \cdot \mathbf{r}_i$, where $\mathbf{f}_i$ is the force on the i^{th} particle whose position is $\mathbf{r}_i$. We consider the time average of a quantity A by

$$\langle A\rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, A(t).$$

(a) Using Newton's second law, show that $\langle\mathcal{V}\rangle = -2\langle K\rangle$, where K is the total kinetic energy of the system. In your proof, you will need to show that the average of the time derivative of a certain quantity is zero; a hint is that this quantity is bounded above and below. Say when this is true, and try to prove the theorem as carefully as you can, using this fact. If you are unable to obtain a proof algebraically, try and argue graphically why the theorem holds. Explain why thermal equilibrium is not necessary for the virial theorem to be valid.

(b) Consider an isolated cluster of N galaxies. Show that the mean potential energy is minus twice the mean kinetic energy. Suggest how this result might be useful to astronomers.