Physics 504: Statistical Mechanics and Kinetic Theory
HOMEWORK SHEET 5

Due 5pm Thur 2 April 2020 in the 504 box.

In this homework, you are asked to sketch certain functions. Please note that sketch means that you do NOT plot using mathematica or Matlab, but that you use your analysis skills to figure out the asymptotics of the functions, limiting forms, zeros, etc. In other words, estimate what the function looks like without plotting it.

Question 5–1.

(a) Write down the equation of state and number constraint equation for a system of ideal bosons in 3D in the grand canonical ensemble, in terms of the functions

$$b_{5/2}(z) \equiv -\frac{4}{\sqrt{\pi}}\int_0^\infty dx\, x^2 \log\left(1 - ze^{-x^2}\right) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}$$

$$b_{3/2}(x) \equiv z\frac{db_{5/2}}{dz}.$$

Hence show that the high temperature behaviour recovers the results of classical statistical mechanics and derive the lowest order quantum correction to the classical equation of state.

(b) Starting from the grand canonical equation of state and using integration by parts, or otherwise, show that the internal energy $U = \langle E \rangle$ is given by $3pV/2$ for both ideal Bose and Fermi gases, even in the quantum mechanical (non-classical) regime. In the former case, work above the Bose-Einstein condensation temperature.

Question 5–2. This question asks you to investigate Bose-Einstein condensation in three dimensions.

(a) Show that the onset of Bose-Einstein condensation occurs at a critical temperature T_c at a given concentration of Bosons, or at critical volume per particle v_c for given temperature, and express these quantities in terms of the b functions from 5–1.

(b) What is the form of the pressure of a Bose gas, both above and below the Bose-Einstein condensation point, using the notation of the previous question.

(c) Sketch the graph of pressure versus volume per particle v at fixed T above and below v_c. Find how the pressure at the transition varies with temperature, and assuming that the Clausius-Clapeyron equation $dP/dT = L/T\Delta v$ is valid, where L is the latent heat per particle at a first order transition and Δv is the volume change per particle, show that $L = [5b_{5/2}(1)/2b_{3/2}(1)]k_B T$. Be explicit about any assumptions that you make and justify them.

(d) Now we will check that the above result is correct by calculating the entropy change per particle ΔS during Bose-Einstein condensation directly, and using the definition of latent heat: $L = T\Delta S$. From the definition of the grand potential, show that $S = \partial pV/\partial T|_{v,\mu}$. Hence calculate the entropy of a Bose gas below T_c, and from the value of ΔS that follows, check the result above for the latent heat.

Question 5–3.

This exercise is about free bosons in two dimensions.

(a) Replace the sum over states by an integral, perform the integral, and hence find the exact closed form expression relating the number of particles N to the fugacity $z \equiv e^{\beta\mu(T)}$. *Be sure to calculate in your answer explicitly what is the density of states in energy in two dimensions.*

(b) Hence determine $\mu(T)$ and sketch your result.

(c) Calculate the expected fraction of particles in the ground state and in excited states, and hence find the transition temperature for Bose-Einstein condensation in two dimensions.

(d) The function $b_\nu(z) \equiv \sum_{l=1}^{\infty} z^l/l^\nu$ tends to the Riemann zeta function $\zeta(\nu) \equiv \sum_{l=1}^{\infty} 1/l^\nu$ as $z \to 1$, which can be proved to be finite for $\nu > 1$. Show that this implies that for $d > 2$ there is Bose-Einstein condensation for $T > 0$.

(e) Theorists often state a "folk theorem" such as: "For $d \leq 2$, long wavelength fluctuations prevent Bose-Einstein condensation". From your analysis in (d), explain why it can be said that the divergence of $b_\nu(z)$ as $z \to 1$ for $\nu \leq 1$ prevents Bose-Einstein condensation. Explain the thinking behind the folk theorem by writing down the integral for N at $\mu = 0$ in $\mathbf{k}$ space, and show that the contribution of modes near $|\mathbf{k}| = 0$ is divergent for $d \leq 2$.