Physics 504: Statistical Mechanics and Kinetic Theory
HOMEWORK SHEET 4

Due 5pm Mon 12 March in the 504 box.

Please attempt these questions without looking at textbooks, if you can. If you do need to refer to my notes or textbooks, the most effective way to do this is to read the relevant section, and then try the question again without looking at the book/notes.

Question 4–1.

Paradox: How can the chemical potential μ be negative, as it is required to be for ideal bosons? After all, it is the Gibbs free energy per particle, so surely it should be positive? In this problem, resolve this paradox qualitatively by using the equivalent definition

$$\mu \equiv \left.\frac{\partial E}{\partial N}\right|_{S,V}.$$

Here are some steps to guide you in your thinking:

(a) What happens to the entropy of a system when (*e.g.*) at or near zero temperature a particle is added to the system with no change in *energy*? Thus, with what sign of energy should one add a particle to the system in such a way that the *entropy* does not change?

(b) How are the considerations above modified for Fermi systems?

Question 4–2.

(a) By appropriately differentiating our general expression for Ξ show that in the grand canonical ensemble, for free fermions

$$\langle n_i \rangle = \frac{\lambda e^{-\beta\epsilon_i}}{1+\lambda e^{-\beta\epsilon_i}}$$

with λ being the fugacity $e^{\beta\mu}$.

(b) Calculate the Fermi energy ϵ_F for ideal fermions with spin s.

(c) Briefly explain why the $\lambda \gg 1$ limit is the appropriate one for low temperature.

(d) Using the number constraint equation and the leading asymptotic form of the function $f_{3/2}(\lambda)$ for $\lambda \gg 1$ given in lectures, for free fermions with spin s at low temperatures, calculate the explicit expression for the temperature dependence of the chemical potential to lowest non-trivial order in T and show that

$$\mu(T) = \epsilon_F[1 + AT^2 + O(T^3)]$$

where A is a coefficient that you must determine (be careful about the sign). Verify that the correction to the zero temperature value of $\mu(T)$ is indeed small as claimed in lectures.

Question 4–3.

A white dwarf is a star that has used up its supply of Hydrogen. Their content is mostly helium, and their density is $\rho \approx 10^7 g/cm^3$. Their mass is typically a solar mass, i.e. $M \approx 10^{33} g$. Their core temperature is $T \approx 10^7 K$; this corresponds to $1000eV$, indicating that the helium is completely ionised, and the star is a fluid of helium nuclei and electrons, the latter regarded as an ideal Fermi gas.

(a) What is the density of electrons in the white dwarf?

(b) What is the Fermi energy and Fermi temperature?

(c) Is the Fermi gas degenerate? What forces act on the star, to determine its equilibrium size? Which forces can be neglected in a first approximation, and why?

(d) Suppose that there are N electrons in a spherical white dwarf of volume V and radius R, at high enough density that relativistic kinematics must be used. Let m_e be the electron mass. Calculate the ground state energy of the white dwarf, leaving your answer in terms of the dimensionless function $f(z) \equiv \int_0^z dx x^2 \sqrt{1+x^2}$.

(e) Calculate the pressure P exerted by the Fermi gas, leaving your answer in terms of $f(z)$.

(f) Show that the condition for equilibrium of the star implies that $P \propto M^2/R^4$. The value of this formula is that you have already calculated P as a function of M and R. Hence, by eliminating P, one could now calculate the radius as a function of white dwarf mass: $R = R(M)$. This formula can be inverted to give $M = M(R)$. The value of M where $R = 0$ represents the upper limit on the mass of a white dwarf, and is known as the Chandrasekhar limit. What happens if the white dwarf has a mass greater than the Chandrasekhar limit?

Question 4–4.

(a) Write down the equation of state and number constraint equation for a system of ideal bosons in the grand canonical ensemble, in terms of the functions

$$b_{5/2}(z) \equiv -\frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \log\left(1 - ze^{-x^2}\right) = \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}}$$

$$b_{3/2}(x) \equiv z\frac{db_{5/2}}{dz}.$$

Hence show that the high temperature behaviour recovers the results of classical statistical mechanics and derive the lowest order quantum correction to the classical equation of state.

(b) Starting from the grand canonical equation of state and using integration by parts, or otherwise, show that the internal energy $U = \langle E \rangle$ is given by $3pV/2$ for both ideal Bose and Fermi gases, even in the quantum mechanical (non-classical) regime. In the former case, work above the Bose-Einstein condensation temperature.