

Turbulence as a Problem in Non-equilibrium Statistical Mechanics

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Abstract The transitional and well-developed regimes of turbulent shear flows exhibit a variety of remarkable scaling laws that are only now beginning to be systematically studied and understood. In the first part of this article, we summarize recent progress in understanding the friction factor of turbulent flows in rough pipes and quasi-two-dimensional soap films, showing how the data obey a two-parameter scaling law known as roughness-induced criticality, and exhibit power-law scaling of friction factor with Reynolds number that depends on the precise form of the nature of the turbulent cascade. These results hint at a non-equilibrium fluctuation-dissipation relation that applies to turbulent flows. The second part of this article concerns the lifetime statistics in smooth pipes around the transition, showing how the remarkable super-exponential scaling with Reynolds number reflects deep connections between large deviation theory, extreme value statistics, directed percolation and the onset of coexistence in predator-prey ecosystems. Both these phenomena reflect the way in which turbulence can be fruitfully approached as a problem in non-equilibrium statistical mechanics.

Keywords Turbulence · Phase transitions · Directed percolation · Extreme value statistics · Non-equilibrium statistical mechanics · Fluctuation-dissipation theorem · Predator-prey ecosystems

The original version of this article was revised: The equation 23 is incorrect. This has been corrected in this version.

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1 Introduction

Fluid turbulence exhibits two regimes where universal scaling behavior can be found. The most studied of these is fully-developed turbulence, which arises at asymptotically large Reynolds numbers, where there are a host of scaling laws in a wide variety of different flows. The general idea is that these scaling laws are manifestations of some type of critical point at infinite Reynolds number that controls scaling for large and finite Reynolds numbers through what are essentially crossover effects. This perspective seems rather simple, but it permits us to understand experimental data on turbulent pipe flows that date back to 1933.

The other regime is the laminar-turbulence transition, which was first studied scientifically by Reynolds in 1883 [67]. Not until the early 21st century were detailed and sufficiently systematic measurements available to challenge and drive theoretical development. The point here is that this transition is not to be regarded as an outcome of low dimensional dynamical systems theory, but is in fact a genuine non-equilibrium phase transition, exhibiting its own critical point scaling laws that can be measured in experiment and calculated in theory. We will see in fact that this transition is most likely to be in the universality class of directed percolation.

This article, in memory of Leo P. Kadanoff, describes selected recent developments in these areas, from the unifying perspective that turbulence should be approached as a problem in non-equilibrium statistical mechanics. These examples demonstrate the utility of the conceptual framework of non-equilibrium statistical mechanics applied to turbulence, and suggests that there may be other fruitful extensions to explore. We are neither the first nor the only authors to have this perspective; for example, see the book [12] or the work of Ruelle, who applies this perspective to the problem of multi-fractal scaling in turbulence [68]. However, the examples presented here are centered around readily observable phenomena that have not been previously considered in the framework of non-equilibrium statistical mechanics, and make new predictions that have been tested experimentally. Leo Kadanoff himself was especially interested in both phase transitions and turbulence, and some of his most enduring contributions were in these areas. The last detailed conversation the authors held with Leo revolved around these topics, and turned out to be influential in our subsequent work on these topics. Thus we are honored to have this opportunity to pay tribute to his memory with this contribution.

2 Friction Factor of Turbulent Flow in Rough Pipes

Fully-developed turbulence shares many features in common with critical phenomena. They are both characterized by strong fluctuations and power-law scaling [22], and naively do not seem to possess a small parameter that can be used to obtain perturbative results for the difference between the mean field scaling exponents and those found in experiment or numerical calculation. Such things have been known in the framework of turbulence since the time of Kraichnan, Edwards and others [19,50,55,79,91,93].

It is therefore natural to ask why it is that the critical phenomenon problem has been solved, but not the turbulence one. To answer this, we should recall how it was that critical phenomena came to be understood, and what were the crucial steps. The complex history of this problem has been extensively reviewed by the active participants [25,45,46,61,87,90] (see also [11] for historical context and the relationship to renormalization in field theory), but the key steps can be seen by going backwards in time. The breakthrough in the problem is

the 1971 article by Wilson [89], whose very title (“Renormalization Group and the Kadanoff Scaling Picture” — a rare instance of a Physical Review editor allowing a title to refer to an individual) indicates that the renormalization group emerged from the key insights of Kadanoff’s 1966 paper on the Ising model [44]. Kadanoff’s paper, in turn, opens by summarizing Widom’s discovery that the free energy of a system near its critical point is a homogeneous function of the relevant coupling constants [86]. Kadanoff went on to show how Widom’s scaling could arise by constructing the effective Hamiltonian at different scales, and making certain technical assumptions. Although incapable of computing critical exponents, the Kadanoff block spin picture, as it came to be known, truly laid the basis for the complete renormalization group solution to the phase transition problem.

The moral of this story for turbulence is that if we are to look for a renormalization group style framework in which to understand turbulence, the starting question should be: what is the analogue of Widom’s scaling law in turbulence?

2.1 Widom Scaling

In the language of magnetic systems, Widom’s scaling law is the statement that the magnetization M as a function of external field H and temperature T is in fact a function of a single variable:

$$M(H, T) = |t|^\beta F(H/|t|^{\beta\delta}) \quad (1)$$

where β and δ are critical exponents for the order parameter and breakdown of linear response theory at the critical isotherm respectively, reduced temperature $t \equiv (T - T_c)/T_c$ and F is a universal scaling function. This data collapse formula is equivalent to two asymptotic scaling laws near the critical point. The first is the order parameter scaling law

$$M \sim |t|^\beta \quad (2)$$

for $H = 0$ as $T \rightarrow T_c$. The second is the breakdown of linear response theory at the critical point. Normally the induced magnetization is proportional to the externally applied field for sufficiently small H . However at the critical point, this relationship becomes a power-law with

$$M \sim H^{1/\delta} \quad (3)$$

for $T = T_c$. The data collapse formula Eq. (1) in effect connects the macroscopic thermodynamics of the critical point with the spatial correlations at small scales, as can be seen from using the other scaling laws and the static susceptibility sum rule [29]. In order for Eq. (1) to be equivalent to the two asymptotic scaling laws Eqs. (2) and (3), the scaling function $F(z)$ must be a particular power-law function of its argument z for large z , so that for $H \neq 0$ and $t \rightarrow 0$, the vanishing t -dependent prefactor and the diverging t -dependent argument of F “cancel out”, leaving simply the power-law function of H that applies on the critical isotherm.

2.2 Roughness-Induced Criticality: Widom Scaling for Wall-Bounded Turbulence

To find an analogue for turbulent fluid flow in a pipe, we need to first ask what is special about T and H . The reduced temperature controls the distance to the critical point, and H can be thought of as a variable which couples to the degrees of freedom to bias them to be ordered. The turbulent analogue of t could be taken to be the inverse of the Reynolds number, Re . The turbulent analogue of H could be wall-roughness; the logic is that in a smooth pipe, the laminar flow is linearly stable to all Re , but wall-roughness on a scale r can create disturbances that grow downstream and eventually fill a pipe with turbulence. If we

are going to construct an analogue of the critical point, we will need experimental data that systematically cover as many decades of the control parameters Re^{-1} and r/D as possible, where the pipe diameter D has been introduced to non-dimensionalize the wall-roughness.

There is one experiment in the whole history of turbulence which contains enough data to work with, and that is due to Nikuradze, who was associated with Prandtl's laboratory during the 1930's [59]. Nikuradze's experiment, never repeated, extended to $\text{Re} \sim 10^6$, encompassing the crossover between laminar and turbulent flows around $\text{Re} \sim 2000$, where the change from Stokes drag is marked by a sudden increase in drag over a small range of Re around 1000–2000 (sometimes known as the “drag catastrophe”). The experiment also covered one and a half decades in wall-roughness, using the same pipe-flow geometry, and measured the normalized pressure drop ΔP along the pipe as a function of Re and r/D . The pressure drop was normalized to yield the so-called friction factor

$$f \equiv \frac{\Delta P/L}{\rho U^2} \quad (4)$$

where L is the pipe length, ρ is the fluid density and U is the mean flow speed.

Next we should ask about the analogue for the two asymptotic scaling laws near the critical point. The limit $T \rightarrow T_c$ is, from our assumptions, equivalent to $1/\text{Re} \rightarrow 0$, whereas the $H \rightarrow 0$ limit is simply equivalent to $r/D \rightarrow 0$ where D is the diameter of the pipe. The existence of this critical point is sometimes known as “roughness-induced criticality” [30]. The order parameter scaling law applies for $H = 0$ and thus corresponds to the behavior of the turbulent fluid as $r/D \rightarrow 0$. Nikuradze's experiments show that in the turbulent regime, as the wall-roughness diminishes, the friction factor follows further and further along the asymptote

$$f \sim \text{Re}^{-1/4}. \quad (5)$$

This scaling was first observed by Blasius [9], and is the analogue of the order parameter scaling law, Eq. (2). The critical isotherm scaling representing the breakdown of linear response theory corresponds to the behavior when $\text{Re} \rightarrow \infty$. In this limit, the friction factor becomes independent of Re , and follows the so-called Strickler scaling law [80]

$$f \sim (r/D)^{1/3}. \quad (6)$$

These stylized facts can be combined into a single scaling law, following the same scaling calculation described above for magnets. The result is that

$$f\left(\frac{r}{D}, \text{Re}\right) = \text{Re}^{-1/4} F\left(\frac{r}{D} \text{Re}^{3/4}\right) \quad (7)$$

where $F(z)$ is a universal scaling function whose asymptotic behavior at small and large values of its argument are determined by the scaling calculation. This scaling law can be readily tested by replotting Nikuradze's data in the form of $f\text{Re}^{1/4}$ versus $\text{Re}^{3/4} \times r(D)$, and a very encouraging data collapse is found [30]. However, the collapse is not perfect, and it is important to understand why.

2.3 Anomalous Dimensions in Turbulence

Turbulence, just as with critical phenomena, is characterized by incomplete similarity [3, 4]. This term, originally used in the context of similarity solutions to deterministic partial differential equations, means that self-similarity is weakly broken by a variable whose small value with respect to the characteristic scale of the solution is nevertheless not negligible.

In critical phenomena, for example, the correlation length $\xi(T)$ diverges near the critical temperature T_c , so that it becomes much larger than the ultra-violet cut-off ℓ , such as the lattice spacing in solid state physics. Even though $\ell/\xi(T) \rightarrow 0$ as $T \rightarrow T_c$, ℓ itself is not negligible, and in fact it is this fact which is responsible for the existence of anomalous critical exponents whose value differs from that expected on the basis of mean field theory. A clear example is the scaling of the two-point correlation function $G(x-x', T_c)$ at the critical point: on dimensional grounds, its Fourier transform $\hat{G}(k, T_c)$ has to have dimensions of (length)² so that one would expect that $\hat{G}(k, T_c) \sim k^{-2}$. In fact, the scaling obeys $\hat{G}(k, T_c) \sim k^{-2+\eta}$, where η is another critical exponent. The way in which dimensional analysis is preserved is that if we write out the functional dependence more precisely, it is found that $\hat{G}(k, T_c) \sim k^{-2}(k\ell)^\eta$. This is a rather counter-intuitive result because one would naively have expected that ℓ could be neglected in the functional form of $G(k, T_c)$ since it is so much smaller than the correlation length $\xi(T)$, which has in fact diverged at the critical temperature. The fact that $\hat{G}(k, T_c)$ retains a dependence on ℓ , i.e. is of the form $G(k, \ell, T_c)$, in what would otherwise have been a purely self-similar regime is incomplete similarity, or more descriptively, scale interference. Complete similarity then corresponds to the assumption in mean field theory, namely that $\eta = 0$, and correspondingly $\hat{G}(k, T_c)$ is a pure power law with no dependence on ℓ .

In turbulence, the scale interference is a statement about the inertial regime behavior. In the limit of $Re \rightarrow \infty$ the inertial regime is generally thought to describe the dissipationless transfer of energy from one scale to another and the K41 assumption is that this is independent of the large scale of forcing L or the Kolmogorov scale η_K beyond which dissipation sets in. These assumptions uniquely determine the form of the energy spectrum $E(k) \sim \bar{\epsilon}^{2/3} k^{-5/3}$ in the inertial range $2\pi/L \ll k \ll 2\pi/\eta_K$, and obey complete similarity. However, if the assumption of complete similarity is not valid, then $E(k)$ can actually be a function of both k and L (the inertial range is sensitive to the manner of turbulent forcing) or k and η_K (the inertial range is sensitive to the dissipative processes at small scales). These considerations are reflected in Kolmogorov’s refined similarity hypothesis, which assumes that the scaling of the longitudinal velocity difference

$$\delta v_\ell^2 \equiv \langle ([\mathbf{v}(\mathbf{r} + \mathbf{n}\ell) - \mathbf{v}(\mathbf{r})] \cdot \mathbf{n})^2 \rangle \tag{8}$$

with an inertial range length scale ℓ varies as

$$\delta v_\ell^2 \sim \ell^{2/3+\eta}. \tag{9}$$

In this expression, η is the intermittency exponent, which arises as an anomalous scaling exponent characterizing the average dissipation over a neighbourhood whose dimension is ℓ .

To see how incomplete similarity modifies the scaling law for the friction factor, we present an argument due to Mehrafarin and Pourtolami [57], that is in the spirit of Kadanoff’s block spin construction [44]. The friction factor is assumed to depend on δv_ℓ and the mean flow speed in the pipe U through a decomposition of the Reynolds stress

$$\tau_R \sim \rho \delta v_\ell U, \tag{10}$$

where ρ is the fluid density. The wall roughness r sets the scale for the transfer of momentum to the wall, so we set $\ell = r$, and the friction factor becomes

$$f \sim \delta v_r / U. \tag{11}$$

Due to the assumed anomalous scaling of the longitudinal velocity difference, we see that the friction factor transforms under a scale transformation of the roughness elements $r \rightarrow \lambda r$ as

$$f \sim r^\alpha \quad \alpha \equiv \frac{1}{3} + \frac{\eta}{2}. \quad (12)$$

The same scale transformation will affect the Reynolds number also, because on the scale of the roughness, the viscosity will scale as $r\delta v_r$, leading to the transformed Reynolds number

$$\text{Re} \rightarrow \lambda^{-(\frac{4}{3} + \frac{\eta}{2})} \text{Re} \quad \text{as } r \rightarrow \lambda r \quad (13)$$

These results show that the friction factor is a generalized homogeneous function, with

$$f\left(\lambda \frac{r}{D}, \lambda^{-(1+\alpha)} \text{Re}\right) = \lambda^\alpha f\left(\frac{r}{D}, \text{Re}\right) \quad (14)$$

Setting the arbitrary scale factor $\lambda \propto \text{Re}^{1/(1+\alpha)}$ we obtain the generalization of Eq. (7) in the form

$$f\left(\frac{r}{D}, \text{Re}\right) = \text{Re}^{-(2+3\eta)/(8+3\eta)} F\left(\frac{r}{D} \text{Re}^{6/(8+3\eta)}\right) \quad (15)$$

where the universal scaling function $F(z)$ behaves as z^α for large z and tends to a constant for small z . A subtlety of this derivation is that the Kolmogorov scale itself varies in a way that depends on the intermittency exponent:

$$\eta_K \sim \text{Re}^{-6/(8+3\eta)} \quad (16)$$

In order to determine the exponent η , Nikuradze's data are plotted analogously to [30], but generalized according to Eq. (15), and the value η adjusted to optimize the data collapse. The resulting value, $\eta = 0.02$ is consistent with previous spectral estimates based on directly measuring the velocity fluctuations and determining $E(k)$. The result is rather remarkable: eight years before Kolmogorov was to formulate the central scaling law of the mean field theory of turbulence, Nikuradze had measured the anomalous scaling exponent merely by accurate measurements of the pressure drop along a turbulent pipe!

The argument presented above [57] is truly in the spirit of Kadanoff's block spin construction, because of Eq. (14). Under a scale transformation, the friction factor as a function of its two arguments retains *the same functional form*, but the arguments get scaled in particular ways. This is analogous to the way in which Kadanoff derived a functional equation for the Helmholtz free energy [44]. He assumed that under scale transformation, the Hamiltonian of a spin system retained its functional form, but the spin degrees of freedom were scaled in a particular way, to take into account that microscopic spins transformed into block spins. Kadanoff's assumption generates the homogeneous functional form of the free energy per spin, and leads to Widom's scaling relations. However, the Kadanoff block spin picture is not capable of computing the actual critical exponents. The reason is that Kadanoff's assumption that the functional form is invariant is wrong in general, because coarse-graining introduces new non-local couplings in the effective Hamiltonian governing the coarse-grained degrees of freedom. Thus the Hamiltonian changes during coarse-graining, and it was Wilson's great achievement to recognize that this can be taken into account by writing down recursion relations for the way in which the coupling constants vary under coarse-graining. Moreover, if these recursion relations flow to a fixed point, then the functional form of the Hamiltonian is invariant (by definition) at the renormalization group fixed point, and in the neighbourhood of this fixed point, the critical exponents can be obtained from a linearization of the coarse-graining transformation.

How could we in principle extend Goldenfeld, Mehrafarin and Pourtolami’s arguments to obtain a genuine RG calculation of the anomalous dimensions of turbulence? The answer must be that one can emulate Wilson’s argument for at least this wall-bounded turbulent shear flow, by obtaining an approximate formula for the way in which the friction factor changes under scale transformations of the roughness, e.g. using decimation in one dimension along the pipe. However, this program has not been completed to date, because it would require a detailed calculation of the transformation of the friction factor under coarse-graining. In fact, even a calculation of the exponents from Eqs. (5) and (6) without anomalous dimensions is only possible using heuristic momentum balance arguments due to Gioia and Chakraborty [27], which we briefly summarize.

2.4 Spectral Link and a Fluctuation-Dissipation Theorem for Turbulence

The starting point is the decomposition of the stress, Eq. (10), where the length scale ℓ is either r or the Kolmogorov scale η_K , depending on whether wall roughness or molecular viscosity makes the greater contribution to the dissipation. Crudely we can represent this by writing

$$\ell = r + a\eta_K \tag{17}$$

where a is a constant of order unity. In Eq. (11), we estimate δv_ℓ by using the definition of the energy spectrum as

$$E(k) \equiv \frac{d}{dk} \left(\frac{1}{2} \delta \hat{v}_k^2 \right) \tag{18}$$

so that

$$\delta v_\ell = \sqrt{\int_{2\pi/\ell}^{\infty} E(k) dk} \tag{19}$$

This leads to the remarkable formula

$$f \propto \sqrt{\int_{2\pi/\ell}^{\infty} E(k) dk} \tag{20}$$

which explicitly connects the velocity fluctuations at small scales with the large scale dissipation f . In this sense, Eq. (20) is a sort of fluctuation-dissipation theorem, establishing the explicit sense in which we can say that turbulence can be usefully understood as a non-equilibrium steady state.

If we use the K41 form

$$E(k) \propto k^{-5/3} \tag{21}$$

and the fact that, in the absence of intermittency, the Kolmogorov scale $\eta_K \sim Re^{-3/4}$ (from Eq. (16)), we find that $f \sim (r/D)^{1/3}$ for large Re , when the important dissipation scale is $\ell \sim r$. This is just the Strickler scaling, Eq. (6). On the other hand, at smaller values of Re , but still in the turbulent regime, the dissipation from wall roughness is insignificant compared to that arising from the cascade to small molecular viscosity scales, and $\ell \sim \eta_K$. This leads to $f \sim Re^{-1/4}$, nicely recovering Eq. (5).

The predictions of this approach are experimentally testable, because the friction factor scaling with Reynolds number (and wall-roughness) is determined precisely by the energy spectrum. So in this statistical mechanical approach, the macroscopic dissipative behavior, as quantified by the friction factor, reflects the nature of the turbulent state through the energy spectrum functional form. On the other hand, the standard theory of wall-bounded

turbulent shear flows is not able to make such a connection. The connection between the energy spectrum and the macroscopic flow properties in steady-state turbulent flows has been termed the spectral link [28,48,83,94].

The spectral link predictions have been tested experimentally in two-dimensional soap films, where turbulent energy spectra with exponents $-5/3$ and -3 can be created, consistent with an inverse energy cascade and a forward enstrophy cascade respectively. The spectral link theory predicts that the friction factor, f , in the smooth-wall Blasius regime should scale with Reynolds number, Re , as $f \sim Re^{-\alpha}$ with $\alpha = 1/4$ and $1/2$ respectively for the inverse energy and forward enstrophy cascades. These results were indeed obtained in careful experiments [83]. At the present time, there are no direct experimental results to test the Strickler exponents. However, direct numerical simulations of the flow in two dimensions with rough walls have been performed by using a conformal map to transform the flow domain into a strip, and then using spectral methods on the resulting transformed Navier-Stokes equations in a strip [36]. These simulations demonstrated energy spectra consistent with both the forward enstrophy cascade and the inverse energy cascade, and the corresponding friction factor scalings were consistent with the predictions of the momentum transfer theory, and exhibited roughness-induced criticality up to $Re = 64,000$.

Sometimes it is objected that turbulence in two dimensions is not the same as it is in three dimensions, because of the absence of vortex stretching. This is of course true, but irrelevant to the perspective expressed here. What is important is the presence of a cascade, regardless of the precise mechanism from which it emerges. The statistical properties of the velocity fluctuations at small distances compared to the integral scale determine the large-scale dissipative processes in turbulence fluids, and the use of statistical mechanical reasoning generates new conceptual insights, such as the idea that friction factors should depend on the energy spectrum, rather than a mean velocity profile as in the standard theory of wall-bounded turbulent shear flows. Statistical mechanics also generates new experimental predictions, which have been partially tested in turbulent soap films.

Despite these advances, there is a lot that needs to be done. The momentum transfer argument for the calculation of the friction factor functional form is too simple, and in particular does not distinguish between streamwise and transverse correlations. The expression Eq. (10) is a crude Reynolds decomposition of the interaction between turbulence and the mean flow, and omits many important details, some of which could perhaps now be addressed at least in two dimensional flows [23]. The connection between the macroscopic dissipation and the microscopic velocity fluctuations arises in a rather ad hoc fashion, and although it expresses the same connection as described by the fluctuation-dissipation theorem, a more formal analysis of the connection would illuminate the way in which turbulence is a non-equilibrium steady state, exhibiting fluctuation-dissipation theorem properties in a way that makes contact with fluctuation theorems far from equilibrium [37,64,73].

3 Statistical Mechanics of the Transition to Turbulence

We turn now to the remarkable scaling behavior near the onset of pipe turbulence around Reynolds number 2000 [1,40], which is now convincingly established as exhibiting the scaling behavior of a well-understood non-equilibrium phase transition: directed percolation (DP).

Briefly, DP is a lattice model of a contact process, with a preferred direction. In the variant known as bond directed percolation, with probability p , bonds are placed on a diamond lattice

oriented at 45 degrees to the preferred direction, and the resulting percolation cluster is grown along the preferred direction, typically starting from a single site, for example. It is found that below a critical value p_c , the cluster will eventually stop growing, so that there will not be a path which percolates through the system. Above p_c , a percolating path can be found, and the point $p = p_c$ exhibits scaling behaviour similar to that found in equilibrium critical points. However, the directed percolation cluster exhibits anisotropic scaling, with different correlation lengths along and perpendicular to the preferred direction. These correlation lengths diverge at p_c , but with different exponents ν_{\perp} and ν_{\parallel} . For a thorough introduction and review of DP, see (e.g.) [39].

3.1 Lifetime of Turbulent Puffs

The transition to turbulence in pipes was of course originally studied by Reynolds [66] but it would be 130 years before measurements of the statistical behavior of the lifetime of turbulence could be performed in a stable and systematic way. Here we describe the recent progress in a selective way to focus on the non-equilibrium statistical physics aspects of the problem. Other recent accounts summarize additional aspects of the problem [6, 17, 54, 63, 78].

The breakthrough measurements in 2006 [41] revealed a surprising and unanticipated result: the so-called mean lifetime τ of turbulence fluctuations (puffs) about an initially laminar flow state increases rapidly as a function of Re , but there is no apparent transition or vertical asymptote. Indeed, it seemed that the data could be represented to a good approximation by

$$\ln \tau \propto Re. \quad (22)$$

These findings led its authors to speculate that the phenomenon of turbulence was in fact just a very long-lived transient state, and that there was neither a sequence of bifurcations [51] nor a sharp transition between laminar and turbulent flows, as had been previously believed. This was especially surprising because for several decades, it had been expected that the transition to turbulence followed the pattern firmly established for the routes to chaos, in particular classic work by Ruelle and Takens on strange attractors [69] and Feigenbaum on period doubling [24]. These works on low-dimensional dynamical systems have influenced recent attempts to describe turbulence using the language and techniques of dynamical systems theory [6, 17, 78]. This has yielded many interesting insights into, and even experimental measurements of, deterministic spatially-localized and unstable exact solutions of the Navier-Stokes equations; but by its nature this approach is less well-suited to explaining the statistical properties of the transition which concern us here.

The early results suggesting that the mean lifetime scales as $\exp(Re)$ were superseded in 2008 by a remarkable *tour de force* [40], which established that the divergence is actually a double exponential (super-exponential) function of Re :

$$\ln \ln \tau \propto Re, \quad (23)$$

with scaling observed over six decades in decay rate $1/\tau$. The functional form $\exp(\exp(Re))$ was argued to arise in some way as a low-dimensional chaotic supertransient [16, 82] but the manner in which the system size was replaced by the Re in these models, and precise details of how this could arise and connect to the Navier-Stokes equations are unclear.

3.2 Extreme Value Statistics

An alternative approach to interpreting the super-exponential behavior is based on the fact that the laminar state is an absorbing state, into which patches of turbulence will decay in the aftermath of a large enough spatial fluctuation in turbulent intensity [72,88]. The decay rate of turbulence is then proportional to the probability that the largest fluctuation overcomes the Re -dependent threshold for the turbulence-laminar transition [31]. By virtue of involving the largest fluctuation, this probability is calculated from the appropriate extreme value distribution [26,35].

Extreme value theory answers the following question. Given a set of independent, identically distributed random variables x_i ($i = 1 \dots N$), what is the probability distribution of the maximum $X_N \equiv \max\{x_i\}$? Unlike the central limit theorem, which provides the unique answer (in most circumstances) to the question of what is the probability distribution of the mean of the random variables (i.e. the normal distribution), the extreme value theorem has three possible answers, depending on the asymptotics of the probability density governing the original variables x_i . For most cases, where this density decays rapidly enough at infinity (as an exponential or faster), the appropriate probability density is the Type I Fisher-Tippett distribution, sometimes also known as the Gumbel distribution [34]. Its cumulative distribution has the form:

$$F(x) = \exp\left(-e^{-(x-\mu)/\beta}\right) \quad (24)$$

where μ and β are parameters that set the location and scale respectively of the distribution. Using this distribution, and Taylor expanding the probability distribution for the largest fluctuation in Re (since the range of Re over which the transition occurs is small: $1800 < Re < 2000$), the super-exponential form Eq. (23) is recovered [31].

At higher values of Re , the puffs were found not only to decay, but also to split. In the puff-splitting regime, the world-lines of puffs trace out a complex branching pattern, observed in experiment and also highly-resolved direct numerical simulations (DNS). Both the decay rate and the rate of splitting followed super-exponential scaling laws with Re , the former increasing with Re and the latter decreasing. Their crossover at $Re \approx 2040$ is interpreted as the single distinguishing Reynolds number in the transitional region, and is identified as the critical value Re_c [1].

3.3 DP and the Decay of Turbulent Puffs

A separate approach to the problem stems from Pomeau's prescient intuition [62] (but see [63] for a counter-argument!) that the laminar state is an absorbing state, into which patches of turbulence will decay in the aftermath of a large enough spatial fluctuation in turbulent intensity [72,88]. Including the diffusion of turbulence, i.e. the spread of turbulent intensity into nearby laminar regions, suggests that the laminar-turbulence transition is governed by a contact process in the universality class of DP [39]. This conclusion follows because DP is widely believed to be the universality class for any local non-equilibrium absorbing process [33,43].

Pomeau's initial suggestion [62] was followed up by simulations of the damped Kuramoto-Sivashinsky equation, where spatiotemporal intermittency coexists with locally uniform domains in a way that seems reminiscent of DP; in particular, as a control parameter is varied, the equation's order parameter evolves as a continuous process beyond a threshold where it jumps discontinuously through a sub-critical bifurcation [14]. A much later study [5], motivated by a perceptive analogy with excitable media, used a model $1 + 1$ dimensional nonlinear partial differential equation coupled to a tent map. Numerical simulations showed

a similar phase diagram to the experiments in pipe flow turbulence, with laminar, metastable turbulence and spatiotemporal intermittency as the control parameter analogous to Re was increased. Furthermore, the order parameter in the spatiotemporal intermittent phase scaled in a way consistent with the order parameter scaling of DP.

To test the DP scenario in more quantitative detail, and to give an interpretation to the super-exponential behavior, simulations of DP were performed in the geometry of a pipe and emulating the conditions of the experiments [76]. The basic idea is that an occupied site on a lattice corresponds to a turbulent correlation volume, and an empty site corresponds to a laminar region. Starting from a localized puff of “turbulence” and for $p < p_c$, the DP region eventually dies away, whereas for $p > p_c$ it spreads and fills the pipe. The mean lifetime τ of puffs could thus be measured, following the procedure used in experiment [40]. These numerical experiments recapitulated the super-exponential behavior observed in the experiments, and moreover provided an alternative rationale for the super-exponential distribution for the decay rate. In DP, the history of occupied sites in successive time slices traces out a complex network of paths, which ends when the last turbulent or occupied site is reached. Thus the lifetime of turbulence is the length of the *longest* path in the DP simulation, and its probability density would follow extreme value statistics [7,76].

These arguments can be extended to the case where there is puff-splitting (i.e. for $p > p_c$), and simulation results confirm the predicted super-exponential dependence for the splitting rate as well [74]. It may seem surprising that DP itself exhibits a super-exponential scaling law. One might wonder why the timescales do not diverge at the percolation threshold p_c with the appropriate power-law divergence. The answer turns out to be subtle and related to the precise way in which τ for decay and splitting is measured [74]. In fact, it is possible in principle to extract the expected power-law divergences from experimental data [74] even while they appear to show super-exponential behavior and thus no signature of a critical point.

To summarise: experimental data and theory strongly suggest that the laminar-turbulence transition in pipes is in the universality class of directed percolation. Recent experiments on ultra-narrow gap large aspect ratio Couette flow [52] and on channel flow [70] report measurements of the critical exponents and in the case of the Couette flow, even the universal scaling functions.

3.4 Landau Theory for Laminar-Turbulence Transition

How is it possible that a driven fluid flow in a spatial continuum could behave precisely like a discrete lattice model from non-equilibrium statistical mechanics, surely an approximation at best? Such exactitude is unprecedented in fluid mechanics but the underlying explanation rests with the theory of phase transitions [29], of which it is our contention that the laminar-turbulence transition is an example. There it is well-established that universal aspects of phase transitions, such as the phase diagram, critical exponents and scaling functions are all described *exactly* by an effective coarse-grained theory (“Landau theory”) that contains only the symmetry-allowed collective and long-wavelength modes, without requiring excessive realism at the microscopic level of description. Being based on symmetry principles, the individual symmetry-allowed terms in Landau theory do not require detailed derivation from the microscopic level of description. This is fortunate given that there is usually no good, uniformly valid approximation scheme to derive formally and systematically these terms and their coefficients from first principles. This is true in equilibrium phase transitions, and all the more so in the laminar-turbulence transition, which occurs far from equilibrium.

In fact, it is neither necessary nor desirable to derive the coarse-grained effective theory from a microscopic description, because any such derivation would need a small parameter

and would thus only have limited validity due to the analytical approximations made. A familiar example of this situation is that even though the Navier-Stokes equations can be derived from Boltzmann's kinetic equations for gases, such a derivation would imply that the Navier-Stokes equation is only valid for dilute gases. In fact, the Navier-Stokes equations are an excellent description for dense liquids as well, and can be obtained by perfectly satisfactory phenomenological and symmetry arguments. The derivation from Boltzmann's kinetic theory is inherently limited by the regime of validity of the kinetic theory—low density—and this leads to an unnecessarily restrictive derivation of the equations of fluid dynamics. Returning to phase transitions, the reason why an analytical derivation of the coarse-grained theory is unnecessary is that even if the coefficients of the terms could be computed in the order parameter expansion of the Landau theory, they do not come into the exponents or scaling functions anyway, and thus they do not affect the critical behavior. In the case of the transition to turbulence, the strategy then is to construct an effective theory that is valid near the transition. This effective theory would be an exact representation of the critical behavior of the laminar-turbulent transition, and as is often the case, could potentially be mapped into one of the canonical representatives of a known universality class. That universality class turns out to be DP.

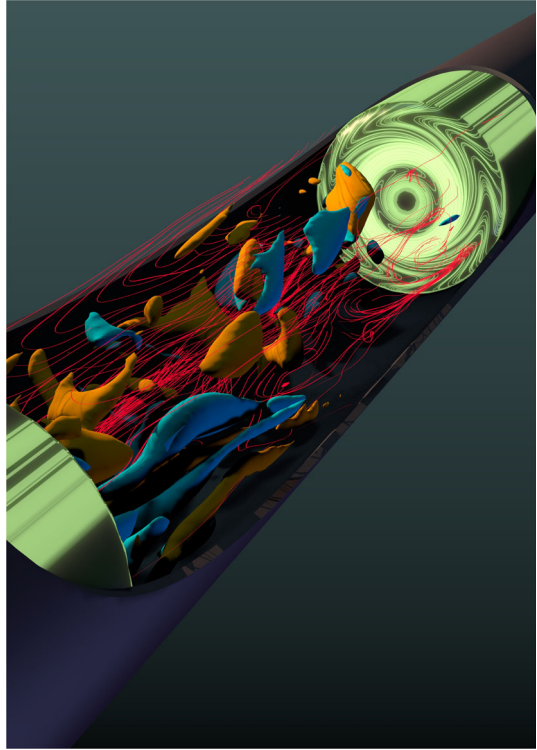
In order to build an effective theory for the transitional turbulence problem by constructing the symmetry-allowed collective and long-wavelength modes, the analytical difficulties are acute. Therefore, to avoid approximations which are difficult to justify systematically, direct numerical simulation was used to identify the important collective modes which exhibit an interplay between large-scale fluctuations and small-scale dynamics at the onset of turbulence, and thence to write down the corresponding minimal stochastic model, in the spirit of the Landau theory of phase transitions [75].

3.5 Zonal Flow and Predator-Prey Dynamics

The result of the numerical simulations was the identification of a collective mode near the transition which regulates turbulence but is itself generated by the turbulence [75]. Technically speaking this mode is a zonal flow. It is purely azimuthal, but has radial and time dependence, and no dependence on axial coordinate z . The mode is not driven by the pressure drop along the axis of the pipe, in contrast to the mean flow. Instead it is driven by the turbulence itself, in particular arising from the anisotropy of the Reynolds stress tensor. The mode shears the turbulence and thus has the effect of reducing the anisotropy of the turbulent fluctuations. In turn, this reduces the intensity of the zonal flow. Once the zonal flow intensity has diminished, the turbulence is no longer sheared so strongly and so is less suppressed than before. As a result the energy in the turbulent modes increases, and the cycle begins again. This narrative of the interplay between zonal flow and turbulence was established by measuring the energy in the zonal flow and turbulent degrees of freedom, the azimuthal flow velocity and the Reynolds stress [75] (Fig. 1).

In general, it is the case that zonal flows are driven by statistical anisotropy in turbulence, but are themselves an isotropizing influence on the turbulence through their coupling to the Reynolds stress [2,60,77]. The interplay between zonal flow suppression of turbulence and turbulence initiation of zonal flow has also been reported in thermal convection in a variety of geometries [32,38]. Originally predator-prey behavior was proposed by Diamond and collaborators [18,42,49] many years ago in the context of the interaction between drift-wave turbulence and zonal flows in tokomaks. The predator-prey oscillations were recently observed in tokomaks [15,20,21,71,92] and in a table-top electroconvection analogue of the L-H transition [2].

Fig. 1 Cutaway view of DNS for pipe flow near the transition to turbulence, showing the zonal flow (*green*), isosurfaces of Reynolds stress (*blue* and *orange*) and streamlines (*red*) (Color figure online)



3.6 Lotka-Volterra Equations in Transitional Turbulence

The activation-inhibition nature of the interplay described here parallels that which occurs in predator-prey ecosystems. Predator-prey ecosystems exhibit the following well-known behavior. A prey acts as a source of food for a predator, and thus the predator population rises. However, under increased predation, the prey population begins to decline. As a result the predator population subsequently declines as well. With reduced predation pressure, the prey population begins to rise, and the cycle begins again. This behaviour is typically modeled by the Lotka-Volterra equations [53,65,85] for the population of predator A and prey B :

$$\dot{A} = pAB - dA \quad (25)$$

$$\dot{B} = bB(1 - B/\kappa) - pAB \quad (26)$$

where time derivative is denoted by a dot, p is predation rate, d is predator death rate, b is prey birth rate and κ is the carrying capacity (i.e. the maximum amount of prey that the ecosystem nutrient supply can support).

An outline of how to derive the predator-prey equations in pipe transitional turbulence is as follows, modeled after efforts to obtain such equations heuristically in tokamak physics [18]. We start by sketching the form of an equation describing the time variation of the energy of turbulent modes, due to local instabilities and the likely interaction with the zonal flow. The basic premise is that there is a primary instability generating turbulence at small scales, probably arising from the interaction of localized unstable modes such as periodic orbits. The energy of turbulent fluctuations E at the relevant wavenumber or range of wavenumbers will

have three main contributions to dE/dt . The first is the primary linear instability of the form $\propto E$. The second term will be of higher order, describing eddy interactions through some sort of non-local scattering kernel or triad processes. We will make the usual ansatz that near a phase transition, it is permissible to replace the non-local kernel by local terms describing eddy damping, of the form $\propto -E^2$. Finally, from the numerics we know that turbulent fluctuations are suppressed by interaction with the zonal flow. What should be the form of this interaction? The zonal flow is a collective shear mode; the azimuthal velocity component u_θ experiences shear in the radial direction r , and will be denoted by $\Omega \equiv \partial \langle \bar{u}_\theta(r) \rangle / \partial r$. Here θ is the azimuthal direction, and $\bar{u}_\theta(r)$ represents the purely azimuthal component of the zonal flow that is spatially uniform in the longitudinal direction, indicating it is not driven by pressure gradients in pipe flows. The damping should occur through interaction between the Reynolds stress and Ω , but should be independent of the direction of the shear. Thus, the most generic coupling between the turbulence and Ω should be proportional to both E and $U \equiv \Omega^2$.

These considerations suggest that

$$\frac{dE}{dt} = \gamma_0 E - \alpha_1 E^2 - \alpha_2 EU \tag{27}$$

where γ_0, α_1 and α_2 are constants.

Next we sketch an outline of how one can obtain a description of the zonal flow equation of motion. The starting point is the Reynolds momentum balance equation, which we will write in the approximate form for the zonal flow velocity field:

$$\frac{\partial \bar{u}_\theta(r)}{\partial t} = - \frac{\partial \langle \tilde{v}_r \tilde{v}_\theta \rangle}{\partial r} - \mu \langle \bar{u}_\theta(r) \rangle \tag{28}$$

where μ is some damping coefficient, the tilde denotes fluctuation component and $\langle \tilde{v}_r \tilde{v}_\theta \rangle$ is the Reynolds stress in the azimuthal direction. In Eq. (28), we have omitted terms that are in principle present from the Reynolds equation, but either vanish due to the azimuthal average or are small compared to the terms retained. We do not have a fully systematic derivation of this equation. However, we have measured the right and left hand terms of this equation in the DNS and the results show that these terms do indeed track one another [75]. Taking the radial derivative of Eq. (28) leads to

$$\partial_t \Omega = -\partial_r^2 \langle \tilde{v}_r \tilde{v}_\theta \rangle - \mu \Omega \tag{29}$$

This equation is not closed of course, but we can make progress with scaling arguments. We conjecture that $\langle \tilde{v}_r \tilde{v}_\theta \rangle$ is quadratic in velocity fluctuations and therefore should be proportional to E . Note that the term $\langle \tilde{v}_r \tilde{v}_\theta \rangle$ vanishes by symmetry in an isotropic flow, but is non-zero when the turbulent fluctuations are anisotropic and coupled to the zonal flow which provides a local directionality to the velocity fluctuations. This suggests that $-\partial_r^2 \langle \tilde{v}_r \tilde{v}_\theta \rangle \propto +\Omega + O(\Omega^2)$ where it is important to note that the $+$ sign means that the Reynolds stress anisotropy is exciting the zonal flow, and not damping it, corresponding to the DNS results. Putting these scaling arguments together and multiplying through by Ω suggests that

$$\partial_t U = \alpha_3 EU - 2\mu U \tag{30}$$

where α_3 is another phenomenological constant. The equation argued for above is basically a scalar equation, but a full understanding of the interaction of mean flows or zonal flows with turbulence anisotropy requires a detailed consideration of the full tensor Reynolds equation, the spatial variation of the eigenvectors of the stress tensor etc. The heuristic derivation described here can be checked by direct numerical computations, in principle, and we hope to do this in the future. The immediate consequence of Eqs. (27) and (30) is that they have

the form of the mean field Lotka-Volterra equations Eq. (25), and thus would be expected to predict (at the mean field level) the existence of predator-prey oscillations.

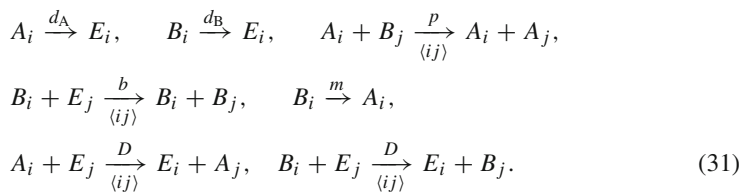
Unfortunately, the mean field Lotka-Volterra equations do not predict population oscillations at all! It is straightforward to see that for finite κ , their steady state is a constant solution, and certainly not a limit cycle. When $\kappa = \infty$, the equations have oscillatory solutions but they are centers, not asymptotically stable limit cycles, and the amplitude and phase depend on the initial conditions. Faced with the paradox posed by the apparent discrepancy between the mathematical solution and the expectation based on the verbal description of predator-prey populations, the standard resolution is to invoke other biological factors such as predator satiation and other forms of what is known as “functional response”. These effects lead to modifications of the Lotka-Volterra equations, introducing nonlinearities that guarantee limit cycle behavior.

The most satisfying resolution of the paradox, however, is that no new biological factors need to be introduced at all: the non-oscillatory prediction from the Lotka-Volterra equations is an artifact of the mean field approximation. As shown by McKane and Newman [56], an individual-level model of predator-prey, wherein each organism’s birth, death and predatory activity is simulated, leads to persistent population oscillations. Stochasticity of individual birth, death, predation processes leads to multiplicative noise in the effective equations at the population level. These equations can be calculated accurately using van Kampen’s system size expansion [47,84]. In the limit of infinite population size, the oscillations vanish, but for finite system size, the oscillations experience a resonance effect which amplifies them. In spatially-extended systems, stochasticity locally drives instabilities, leading to fluctuation-induced Turing patterns or traveling waves if there are appropriate sources of nonlinearity [8,10].

3.7 Stochastic Predator-Prey Model for Transitional Turbulence

These considerations are significant for the interactions between turbulence and zonal flows, as stochasticity, in effect, arises due to the microscopic interactions between localized unstable modes of the fluid flow. The predator-prey nature of the interactions shows that turbulence is the prey, whereas the zonal flow is the predator. In order to construct a Landau theory for the laminar-turbulence transition, it is necessary to write down all possible interactions between the turbulence and the zonal flow. These are summarized in Fig. 2, and displayed in the language of stochastic predator-prey processes.

The equations for the emergence of a zonal flow collective mode interacting via activator-inhibitor/predator-prey kinetics with the small-scale turbulence can be written down as a set of spatially-extended rate equations for the number of predator A and prey B and nutrient (laminar flow) sites (E) as follows:



where d_A and d_B are the death rates of A and B , p is the predation rate, b is the prey birth rate due to consumption of nutrient, $\langle ij \rangle$ denotes hopping to nearest neighbor sites, D is the nearest-neighbor hopping rate, and m is the point mutation rate from prey to predator, which models the induction of the zonal flow from the turbulence degrees of freedom.

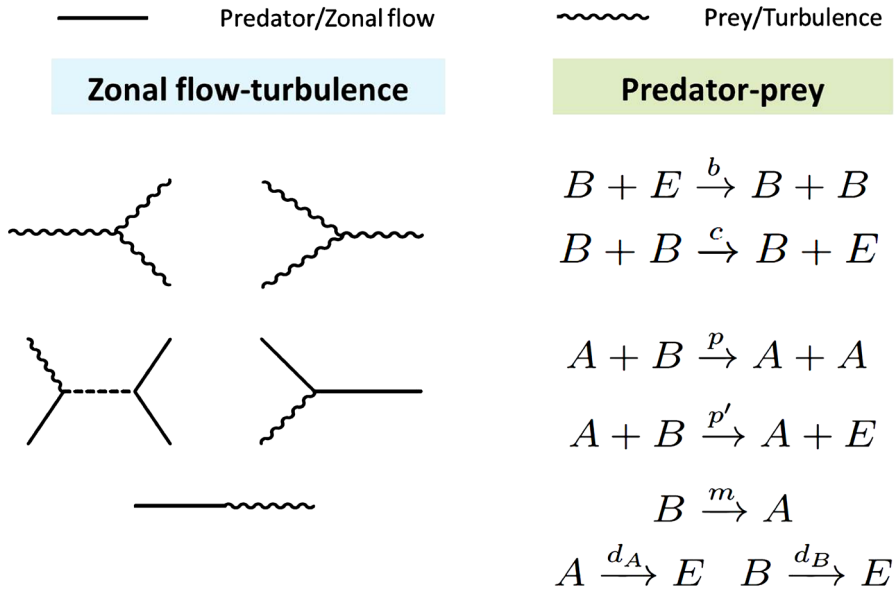


Fig. 2 Effective theory for interactions between small-scale turbulence and large-scale zonal flow. The allowed interactions between turbulence modes (prey B , wiggly lines) and zonal flow (predator A , straight lines) are shown, together with their interpretation as processes describing birth, death and predatory activity. An allowed process at the lowest order corresponds to the conversion between prey and predator with rate m , something that does not have a direct realization in most biological systems. The symbol E denotes the third trophic level in the ecosystem of turbulence, namely the food that sustains the prey, which in the fluid dynamics picture is simply the laminar flow state itself. The symbols above the arrows denote rate constants. The definition of predator-prey processes is described in the text. In the right column the predation process with rate p' is not included in our model since it only renormalizes the predation coefficient in the prey equation in Eq. (25)

Remarkably, simulations of this predator-prey model, in a 2D strip intended to represent the 3D pipe geometry of the original turbulence experiments, reproduce the main features of the laminar-turbulence transition. In this case, the control parameter turns out to be the birth rate b of the prey, and this is the analogue of Re [75]. First of all the phase diagram is reproduced as a function of the birth rate b of the prey, which plays the role of Re . In particular there is a phase where no prey survive; then at higher b , a phase where the prey and predator co-exist, but localised regions of prey decay; then at still higher values of b , a region where localized regions of prey split, so that the dynamics exhibits the strong intermittency in space and time seen in the turbulence simulations and experiments. Furthermore, it is found that there is a super-exponential variation of decay and splitting lifetimes on the prey lifetime b [75].

In addition to recapitulating the phenomenology of the laminar-turbulence transition in pipes, the stochastic predator-prey model Eq. (31) can be mapped exactly into Reggeon field theory [58,81] and this field theory itself has long been known to be in the DP universality class [13,43]. The connection between the super-exponential scaling of timescales with Re and the expected divergence at a critical value of Re is not explained in the original work [75]. In fact, subsequently it has been understood how to extract the dynamic critical exponents and the divergence of lifetimes from the turbulence data, at least in principle [74].

3.8 Summary

In summary, we used DNS to identify the important collective modes at the onset of turbulence—the predator-prey modes—and then wrote down the simplest minimal stochastic

model to account for these observations. This model *predicts* without using the Navier-Stokes equations the puff lifetime and splitting behavior observed in experiment. This approach is a precise parallel to that used in the conventional theory of phase transitions, where one builds a Landau theory, a coarse-grained (or effective) theory, using symmetry principles. This intermediate level description can then be used as a starting point for renormalization group analysis to compute the critical behavior. In this case, however, the statistical description arises from non-equilibrium statistical mechanics, as the predator-prey equations do not obey detailed balance.

Directed percolation arises due to the appearance of collective modes near criticality whose fluctuations exhibit the characteristics of stochastic predator-prey dynamics near the collapse of an ecosystem from its coexistent state. Both turbulence and predator-prey ecosystem criticality reflect scaling laws that ultimately derive from extreme value statistics, thus, establishing an unprecedented connection between the laminar-turbulence transition, predator-prey extinction, directed percolation, and extreme value theory.

Our approach is thus a precise parallel to the way in which phase transitions are understood in condensed matter physics, and shows that concepts of universality and effective theories are applicable to the laminar-turbulence transition.

4 Conclusion

In this article, we have shown with concrete examples how turbulence can usefully be viewed through the lens of non-equilibrium statistical mechanics. In particular we have shown how macroscopic dissipative properties of wall-bounded turbulent shear flows, but especially pipe flow, can be described using concepts from scaling theory for $Re > 2000$. Moreover precise results from the conceptual framework of renormalization group theory were used to identify the universality class of the laminar-turbulence transition. These results are suggestive of a more profound connection between turbulent flows and non-equilibrium statistical mechanics.

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