DYNAMICS OF COSMOLOGICAL PHASE TRANSITIONS: WHAT CAN WE LEARN FROM CONDENSED MATTER PHYSICS?[1]

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Abstract. A brief outline is given of the description of phase transition kinetics in condensed matter systems with a continuous symmetry, emphasising the roles of dissipation, coarse-graining and scaling. The possible relevance of these ideas to the early universe is explored in the contexts of the GUT string transition and the electroweak transition.

1. INTRODUCTION

How fast do phase transitions occur? Remarkably, it is found that although it is often straightforward to estimate a characteristic relaxation time for the microscopic degrees of freedom, the actual characteristic time for completion of the phase transition may be many orders of magnitude greater. For example, laboratory experiments indicate that following a temperature quench, the transition to the superconducting state of a normal metal in a magnetic field may take many minutes. The primary reason for the slowness of the transition is the formation, interaction and subsequent dynamics of topological defects. In systems with a discrete symmetry and a scalar order parameter, such as binary alloys, the topological defects are domain walls, whilst in systems with a continuous symmetry and a vector or tensor order parameter, such as certain liquid crystals, the defects may be strings and monopoles. The motion and mutual annihilation of the defects is usually the rate-determining step for the transition, and is affected by such factors as the presence of dissipation or disorder, the range of the interactions between defects, and even their homotopy classification.

[1]Invited lectures, to appear in the proceedings of the NATO ARW on "Formation and Interactions of Topological Defects", A.C. Davis and R.H. Brandenberger (eds.). Plenum Press., N.Y., 1995.

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P-95-01-002

Such phenomena are ubiquitous in condensed matter, yet only recently have detailed studies been made of phase transition kinetics in condensed matter systems with non-scalar order parameters. Although interesting in their own right, these examples may be regarded as caricatures of the phase transitions that are thought to have occurred in the early universe as it cooled. The principal analogous feature is the spontaneously broken global continuous symmetry and, in superconductors at least, the existence of a local gauge symmetry.

The purpose of this article is to outline briefly the way in which the kinetics may be investigated theoretically, with an eye towards using the techniques of condensed matter physics in a cosmological context, where appropriate. In particular, the work of the group at Illinois on phase ordering in systems with continuous symmetries is relevant. This primarily numerical work includes studies of the non-conserved dynamics of the XY model in two dimensions[1], three dimensions[2], the conserved XY model in one and two dimensions[3], the dynamics of the superconducting transition[4], the dynamics of the Ising gauge theory, where there is no local order parameter[5] and ordering in uni- and bi-axial liquid crystals[6]. Other relevant numerical studies are those of Toyoki[7] and Bray, who has recently given a complete review of the topic of phase ordering in systems with continuous symmetries[8].

Since all of the literature and an extensive review are easily available, it does not seem worthwhile to provide duplication of the results here. Instead, the focus will be on principles and concepts, together with some remarks comparing the procedures used in condensed matter physics with those used in cosmological applications. These are particularly pertinent in the case of the electroweak transition, where the phase transition kinetics may not occur via nucleation and growth, as has sometimes been assumed.

1.1. Scaling

Let us consider qualitatively the sequence of different time regimes exhibited by a binary alloy undergoing spinodal decomposition. First, the small amplitude long-wavelength fluctuations of the order parameter, present from the initial conditions, become amplified exponentially: this behaviour is predicted by the linearised equations of motion, and is rarely observed. The nonlinearities in the equation of motion quickly stabilise the order parameter at its two equilibrium values almost everywhere. Thus a series of domains has been formed, separated by domain walls. This interlocking pattern of domains subsequently coarsens, driven by the excess energy from the curvature of the domain walls. At very long times, the system attains equilibrium, which energetically should simply be a single domain wall dividing the system into two coexisting equilibrium phases. The intermediate time regime, where the domains simply coarsen, exhibits *dynamic scaling*[9].

Dynamic scaling simply means that at large enough times t, the emerging pattern contains only one time-dependent length scale $L(t)$. Thus, the equal time two-point correlation function of the order parameter ψ defined by

$$C(\mathbf{r},t) \equiv \langle \psi(\mathbf{r},t)\psi(\mathbf{0},t)\rangle \tag{1}$$

is actually only a function of a reduced variable:

$$C(\mathbf{r},t) = F(r/L(t)) \tag{2}$$

where F is known as a scaling function. In practice, for finite times, there is a weak explicit time dependence in F, which proceeds to increasingly short distances, whilst $L(t)$ converges to its 'ideal' power law form, as discussed below.

Of course, there are other length scales present: the bulk correlation length for longitudinal order parameter fluctuations $\xi(T)$, which is a function of the temperature T after the quench, and the microscopic length scale a, on the order of molecular dimensions. But for sufficiently long times, the following inequalities hold: $L(t) \gg \xi(T) > a$. Since thermal fluctuations are operative up to the scale of ξ, the unimportance of ξ relative to L is sometimes stated in a suggestive way by saying that phase ordering "is controlled by a zero temperature fixed point". (Despite the intuition that some sort of renormalisation group (RG) approach should form the basis of a general theory for the approach to equilibrium, a predictive RG theory remains elusive.) The scaling regime may be called an *intermediate asymptotic regime*. For example, in a binary alloy, at short times, the initial fluctuations in the system are being amplified by the unstable growth process, and domain walls are formed, whereas for long enough times in a finite system, the system reaches thermal equilibrium, where only one domain wall traverses the system. In both of these regimes, thermal fluctuations are important. For some set of intermediate times, whose duration is an increasing function of the system size, the scaling regime is observed, and thermal fluctuations are not important in the sense described above. Empirically, it is found that the growth of the characteristic scale L follows a power law form:

$$L(t) = A\xi(T)\,(t/\tau(T))^{\phi}, \tag{3}$$

where $\tau(T)$ is the temperature dependent order parameter relaxation time, A is an amplitude assumed to be of order unity (but not yet measured as far as I know), believed to be universal and ϕ is now believed to have the value $1/3$ for alloys or other conserved systems with a discrete symmetry in dimension $d \geq 2$. For systems with a non-conserved parameter in these dimensions, ϕ has the value $1/2$. Eq. 3 is valid near the transition temperature T_c, and a more general form valid for all $T < T_c$ has been proposed by Bray[10].

Heuristic arguments, given elsewhere in this volume (see, for example, the lectures by Bray), connect these power laws with the motion of domain walls. So one is led to ask how these scaling results are affected when the system exhibits a continuous symmetry, rather than a discrete one, so that the domain walls no longer exist. In fact, other topological defects are present in such systems, and can give rise to different growth laws. These considerations formed the motivation for the work done at Illinois.

1.2. Computer simulation

In order to observe quantifiably the scaling regime in a computer simulation, two conditions must be met: the system size needs to be as large as possible, so that the regime lasts as long as possible, and then the longest possible times must be attained. In addition to these requirements, a number of other results are known which are diagnostics of the scaling regime: that is, they are only satisfied in the scaling regime. These are the Tomita sum rule[11], Yeung's law for the $k \to 0$ behaviour of the X-ray scattering form factor at small wavenumbers k[12], and Porod's law for the form factor's short distance behaviour[13]. In the absence of an obvious small parameter, systematic analytical work has rarely been possible — a notable exception is the study of the $O(N)$ model for $N \to \infty$, where there are no topological defects[14] — and much of our knowledge

has come from computer simulations. Thus, in order to estimate accurately (*e.g.*) the exponent ϕ, computationally efficient techniques have been designed to probe as far into the scaling regime as possible: these are discussed in section 3.3. In fact, rather little of the numerical work to date actually satisfies the known criteria for the asymptotic regime; the most complete work to date on the alloy phase separation problem in three dimensions is that of Shinozaki and Oono[15].

2. FORMULATION

2.1. Level of description

The main difficulty in constructing a theory for phase ordering is the disparity between the different length scales present (ξ, $L(t)$ and a), and the complexity of the actual microscopic equations of motion. However, we are mainly interested in phenomena on the scale of L, so a coarse-grained description is adequate. This sort of approach is common in condensed matter physics: for example, the BCS theory of superconductivity is a well-tested microscopic theory but is virtually useless in situations with spatial variation, such as near boundaries or for time dependent phenomena. Instead, the Ginzburg-Landau theory, a phenomenological theory for coarse-grained order parameter, slowly varying on the scale of ξ, is used.

2.2. Coarse-grained order parameter

Let us be more precise, taking as our example, the case of a superconductor. The order parameter Ψ is zero for $T > T_c$ and nonzero for $T < T_c$. It can be defined in terms of an anomalous Green function by

$$\Psi(\mathbf{x}) \propto \left\langle \hat{\psi}_\downarrow(\mathbf{x})\hat{\psi}_\uparrow(\mathbf{x}) \right\rangle, \tag{4}$$

where $\hat{\psi}_\downarrow(\mathbf{x})$ is a down-spin electron field operator and the angle brackets denote an equilibrium thermal expectation value[16]. The normalisation of Ψ may be chosen with a convention that does not concern us here. In the Meissner and normal phases of a superconductor, the system is translationally invariant and Ψ is spatially uniform. Near equilibrium, Ψ may vary in space, but we will only consider the long wavelength variations: in spinodal decomposition, the instabilities occur at wavelengths long compared with a. Conceptually, the coarse-grained order parameter is defined by the long wavelength Fourier components $\tilde{\Psi}_\mathbf{k}$:

$$\Psi_\Lambda(\mathbf{x}) \equiv \sum_{|\mathbf{k}|<\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\Psi}_\mathbf{k} \tag{5}$$

where the coarse-graining scale Λ satisfies $a \ll \Lambda^{-1} < \xi$. We want to coarse-grain as much as possible, but if we coarse-grain beyond the correlation length then we will have two-phase coexistence within one coarse-graining volume. This over-coarse-grained description would then not be able to tell us about the dynamics of phase separation. The coarse-grained free energy governs the effective dynamics of these long wavelength modes, and is obtained by integrating out the short wavelength modes up to the scale Λ:

$$e^{-F_\Lambda\{\Psi_\Lambda(\mathbf{x})\}/k_BT} \equiv \int \prod_{|\mathbf{k}|>\Lambda} d\Psi_\mathbf{k} e^{-H/k_BT} \delta(\Psi_\Lambda(\mathbf{x}) - \sum_{|\mathbf{k}|<\Lambda} e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\Psi}_\mathbf{k}), \tag{6}$$

where H is the Hamiltonian and k_B is Boltzmann's constant. Thus, for a given coarse-grained order parameter profile in space, a value for the coarse-grained free energy can be calculated. In thermal equilibrium, the probability that the coarse-grained order parameter has a given profile $\Psi_\Lambda(\mathbf{x})$ is proportional to $\exp(-F_\Lambda\{\Psi_\Lambda(\mathbf{x})\}/k_BT)$. The explicit calculation of F_Λ is technically complicated, and unnecessary to perform if one is interested in universal equilibrium quantities. If one is interested in computing the dynamics, certain aspects may not be universal, and will depend upon F_Λ. A recent example where F_Λ has been explicitly calculated using the RG is the work of Alford and March-Russell[17].

It is important to notice the distinction between the calculation of F_Λ and the finite temperature effective action S of field theory. The latter involves no notion of coarse-graining (*i.e.* $\Lambda = \infty$), and is rigorously convex. On the other hand, there is no requirement of convexity for the coarse-grained free energy, and indeed, it has the familiar double-well or wine-bottle form for $T < T_c$. The non-convexity of the one-loop effective action is an artifact of the loop expansion.

2.3. Dynamics

How should we describe the dynamics of the phase transition? Ideally one would solve for the time-dependent density matrix, using the exact microscopic Hamiltonian. Observables such as the coarse-grained order parameter, or its correlation functions, could then be computed. This would involve averaging over the initial conditions with the appropriate Boltzmann weight, and modelling the quench itself. In other words, one would compute the exact dynamics, then coarse-grain. An alternative procedure would be to find the effective equations of motion for the coarse-grained order parameter by directly coarse-graining the equations of motion for the density matrix. A plausible substitute is to write down the phenomenological equation of motion for Ψ_Λ, assuming that the driving force is F_Λ. Even this is rarely done, because usually F_Λ is not known. Instead, a phenomenological form of F_Λ is used.

A phenomenological Langevin equation for Ψ_Λ is obtained by assuming that the force driving the system towards thermal equilibrium is proportional to the deviation from equilibrium:

$$\tau_0 \partial_t \Psi_\Lambda = -\frac{\delta F_\Lambda}{\delta \Psi_\Lambda^*} + \eta \tag{7}$$

where we have assumed that Ψ_Λ is a complex scalar field (appropriate for a superfluid, for example), and included a thermal noise term η so that the system is guaranteed eventually to attain the global minimum of F_Λ. We shall refer to eq. 7 as the time-dependent Ginzburg-Landau equation (TDGL). As mentioned above, the noise term is believed not to play a significant role during the scaling regime. The relaxation time τ_0 is in principle calculable from a microscopic theory (such as BCS theory in the case of a superconductor). The Langevin equation above does not include any conservation laws, although these are easy to include when required. For a superconductor, an appropriately generalised local gauge invariant time-dependent Ginzburg-Landau description can be written down (see *e.g.*, [4]).

It is difficult to assess the regime of validity of eq. 7. Usually it is regarded as a minimal model of phase ordering kinetics, in the sense that it correctly predicts all universal phenomena. However, it may not be accurate for other quantities, and in this sense, it is sometimes said that eq. 7 is a semiquantitative description. An important point is that we have assumed that the dynamics is purely relaxational. Whilst this

is expected to be valid near a critical point, where the correlation length is large, for quenches to low temperatures this is not necessarily the case.

Let us use again our example of the coarse-grained dynamics of a superconductor to see what can go wrong. There, the microscopic BCS theory may be formulated as a set of coupled equations for the Green's functions of the theory, which in turn can be reexpressed as an integral equation for the order parameter[16]. The TDGL is obtained by expanding the integral equation in powers of wavenumber k and frequency ω. However, only at low temperatures, near the critical temperature or when there is strong scattering from impurities does this procedure yield a result independent of the *ratio* ω/k, leading to a *local* partial differential equation. In all other situations, there is no possible local description of even the long wavelength, low frequency order parameter dynamics. Furthermore, at low temperatures, it turns out that the dynamics is not overdamped but exhibits wave-like solutions[18].

3. NUMERICAL METHODS

3.1. Monte Carlo Simulation

This is probably the most straightforward conceptually. Here, the Monte Carlo time is assumed to be proportional to real time, and the dynamics that the system undergoes is assumed to be somehow similar to that of the real system. Of course, this may not be the case: Monte Carlo simulation is only a stochastic process which samples the correct equilibrium distribution. However, a judiciously chosen Monte Carlo dynamics may well be a reasonable caricature of the actual dynamics. This method is usually very slow, and it has not proved to be reliable in extracting the correct long time behaviour. It is not the method of choice nowadays.

3.2. Molecular Dynamics for the Defects

As discussed at length in this volume, topological defects are created during the phase transition, and equilibrium may only occur after all such defects have annihilated. In many situations, the potential for interactions between defects may be calculated, and the defects treated as classical particles, subject to damping and the inter-defect potential. Accordingly the equation of motion for the particles may be readily solved. An example, in the case of superconductor dynamics, is ref. [19]. This approach is relatively fast in terms of computer time, and large systems may be straightforwardly treated. The disadvantage, however, is that it is usually non-trivial to obtain the correct equation of motion for the defects, either because the potential may be hard to obtain, or because the damping may be difficult to calculate. An interesting example is the calculation of ordering in the non-conserved two dimensional XY model in ref. [20]. Here, the ordering proceeds via the annihilation of $\pm$ vortices, whose interaction potential U is logarithmic in their separation R. The equation of motion for the separation R of two vortices is assumed to be

$$\nu \frac{dR}{dt} = -\frac{dU}{dR} \tag{8}$$

where ν is a damping coefficient. This calculation, as well as the molecular dynamics simulation with many vortices, finds that the average separation L varies as $L \sim t^{1/2}$, whereas the correct result is now believed to be $L \sim (t/\log t)^{1/2}$: the logarithmic factor

arises because the damping coefficient ν is actually logarithmically dependent on R [21]. Note that the above estimate of scaling exponents, based upon the overdamped equation of motion for a pair of topological defects, works well in other situations, even correctly predicting the crossover in a superconductor with penetration depth λ from $L \sim (t/\log t)^{1/2}$ $(L < \lambda)$ to $L \sim \log t$ $(L > \lambda)$[4].

3.3. Order Parameter Evolution - PDEs and CDS

Probably the most effective way to explore phase transition kinetics is to solve directly the TDGL or equivalent partial differential equation (PDE). This involves discretising the PDE on a space-time lattice:

$$\Psi(\mathbf{x}, t) \rightarrow \Psi(\mathbf{n}\Delta x, i\Delta t) \equiv \Psi^i_{\mathbf{n}} \tag{9}$$

and using a time-stepping algorithm such as the explicit Euler scheme

$$\Psi^{i+1}_{\mathbf{n}} = \Psi^i_{\mathbf{n}} + \Delta t \left[-\frac{\delta F}{\delta \Psi^*} \right]^i_{\mathbf{n}}. \tag{10}$$

Of course, eventually the continuum limit must be taken: Δx, $\Delta t \rightarrow 0$, subject to possible stability criteria giving an upper bound to $\Delta t/\Delta x^2$. The disadvantage of this method is primarily that of speed and memory: to explore asymptotically large times, with a time step tending to zero requires many iterations of eq. 10.

The cell dynamic system (CDS) method[22] exploits universality in order to overcome this problem, and is now the most widely used approach in studying phase ordering and other pattern formation problems in condensed matter. The basic idea is the observation that it is rather wasteful to model the phenomenon in question by a PDE, which one must then discretise to obtain a set of coupled maps (such as eq. 10) suitable for numerical computation. Instead, the phenomenon is modelled directly by a set of coupled maps, defined on a coarse-grained space-time lattice, with spatial cells of dimension of order $2\pi\Lambda^{-1}$.

To illustrate the basic idea, consider the case of an order-disorder transition, where the order parameter is a simple non-conserved real scalar field obeying the TDGL, and equilibrating with the coarse-grained free energy

$$F\{\Psi(\mathbf{x})\} = \int d^d\mathbf{x} \left[\frac{1}{2}(\nabla\Psi)^2 + \frac{a}{2}\Psi^2 + \frac{b}{4}\Psi^4 \right]. \tag{11}$$

The parameter a changes sign at T_c, is initially positive when $T > T_c$, and at $t = 0+$ is supposed to become instantaneously negative. When substituted into the TDGL, a nonlinear PDE is obtained. The CDS approach need make reference neither to the TDGL nor to the coarse-grained free energy functional of eq. 11. Let us consider the dynamics of the order parameter in one cell, throughout which the order parameter value is essentially constant, but allowed to vary in time. The time dependence of Ψ will have three fixed points, two symmetrically placed about one at 0, corresponding to the three extrema at $\Psi = 0, \pm\sqrt{a/b}$. The former is an unstable fixed point of the cell dynamics, whereas the other two are stable fixed points. We can model the cell dynamics by *any* map $\mathcal{M}_{\Delta t}$ that has this fixed point structure. Thus

$$\Psi^{i+1}_{\mathbf{n}} = \mathcal{M}_{\Delta t}\{\Psi^i_{\mathbf{n}}\}. \tag{12}$$

It is often convenient to take the map

$$\mathcal{M}_{\Delta t}\{\Psi\} = 1.3\tanh(\Psi), \tag{13}$$

although even simpler piece-wise linear maps have also been used. If desired, one could explicitly calculate $\mathcal{M}_{\Delta t}$ from the TDGL, although there would be no point for present purposes: the phenomenological differential equation or phenomenological PDE have no more privileged status than our phenomenological map. The interaction between cells should reflect the role of diffusion processes, and should be as isotropic as possible. Since diffusion is simply a local averaging procedure, we couple the cells in the final form of our CDS by writing

$$\Psi_{\mathbf{n}}^{i+1} = \mathcal{M}_{\Delta t}\{\Psi_{\mathbf{n}}^{i}\} + D\left(\left\langle\Psi_{\mathbf{n}}^{i}\right\rangle - \Psi_{\mathbf{n}}^{i}\right) \tag{14}$$

where D is a phenomenological coefficent, akin to a diffusion constant (but incorporating the spatial and time discretisation units) and the averaging operation is in (*e.g.*) two dimensions

$$\left\langle\Psi_{\mathbf{n}}^{i}\right\rangle \equiv \frac{1}{6}\sum_{n.n.}\Psi_{\mathbf{n}}^{i} + \frac{1}{12}\sum_{n.n.n.}\Psi_{\mathbf{n}}^{i}. \tag{15}$$

Here n.n. means nearest neighbours, n.n.n. means next nearest neighbours. This form of Laplacian is more isotropic than the conventional discretisation[15, 23], as can be seen by examining isocontours of its Fourier-transform.

Empirically, this CDS modelling and its extensions works very well in a wide variety of situations, and gives results that in some cases have been compared with direct integration of the TDGL. This is not surprising: the actual form of the map used is of little consequence for large scale structure, and only influences the detailed form of the order parameter profile near a domain wall or topological defect. Of course, since the free energy of eq. 11 is only phenomenological, its predictions for this variation are no more reliable than those of the CDS map.

The key point about the CDS method, and one that is frequently misunderstood, is that there is no continuum limit: Δx and Δt are not infinitesimals. Thus, very rapid simulations are possible on large systems. The correct way to view the CDS map is that one has integrated or coarse-grained some microscopic equations of motion up to the space-time scale of interest, thus obtaining the map we have guessed phenomenologically. This is conceptually different from the usual approach of numerical analysis, which is to sample a PDE on a sequence of finer and finer meshes. There is no proof that the CDS algorithm is in the same universality class of the PDE (whatever that may be), but the important point is that both are in the same universality class as the physical phenomenon of interest. This is the job of the physicist: to identify and characterise such universality classes. Ref. [24] describes recent work on the application of RG to extract universal features of PDEs.

In the literature, a popular variant on the CDS method is often encountered, in which the conventional Euler discretisation of a PDE is used, but with large Δt and Δx. This works well too, although one must then be careful not to identify the potential used in this discretisation of the PDE with the potential in the original continuum limit PDE, for the reasons described in the preceding paragraph. The tanh map advocated above is simply a convenient form of the potential, which happens to avoid a secondary numerical instability by virtue of being injective.

Computer simulations and experiments on liquid crystals very clearly show that following an instantaneous quench from above T_c, the order parameter fluctuates wildly with position, and the order parameter configuration is so disordered that it is meaningless to identify topological defects. After all, a topological defect is a defect configuration in a smooth, ordered background, and when this does not exist, individual defects cannot be distinguished. This is equivalent to saying that the cores of the defects overlap. Thus, right after the quench, the initial string density ρ_i is not well-defined. In fact, it first becomes well-definable at a time t_1 when the average separation $R(t_1)$ between defects is greater than the core size, given by the order parameter correlation length $\xi(T_f)$ at the final temperature T_f reached by the quench. Hence, the "initial" string density (meaning the string density at this time) can only be given by

$$\rho_i = 1/R(t_1)^2 \sim \xi(T_f)^{-2}. \tag{17}$$

Note that the string correlation function at this and subsequent times, which will depend on the quench history, as well as the static correlations in the order parameter from the starting temperature above T_c.

4.3. Electroweak transition

There is currently considerable interest in the dynamics of the electroweak transition, due to the suggestion that baryon asymmetry arises in the propagating bubble walls accompanying this putative first order transition. For a review, see the article by Turok in this volume. The electroweak transition is conceptually similar to the superconducting transition when gauge field fluctuations are included[27], and we shall couch our discussion in these terms.

The equilibrium statistical mechanics near the superconducting transition is obtained from the partition function

$$Z = \int D\Psi\, D\mathbf{A}\, e^{-F\{\Psi,\mathbf{A}\}/k_BT} \tag{18}$$

where the coarse-grained free energy is usually taken to be

$$F\{\Psi,\mathbf{A}\} = \int d^dx \left[\frac{1}{2}|D\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{(\nabla\times\mathbf{A})^2}{8\pi}\right]. \tag{19}$$

Here $\mathbf{A}$ is the electromagnetic vector potential, $D\Psi \equiv \nabla - ie\mathbf{A}$ is the covariant derivative, and we have worked in the gauge where the scalar potential is zero and $\nabla\cdot\mathbf{A} = 0$. Halperin, Lubensky and Ma[27] showed that the functional integral over the gauge field leads to a term in the resultant coarse-grained free energy density

$$\tilde{F}\{\Psi\} = -k_BT \log \int D\mathbf{A} \exp(-F\{\Psi,\mathbf{A}\}/k_BT) \tag{20}$$

which is proportional to $-|\Psi|^3$, and thus generates a first order transition, which generically proceeds by nucleation and growth of bubbles. For strong supercooling, the nucleation barrier vanishes, and the dynamics proceeds by spinodal decomposition. In the cosmological context of the electroweak transition, it is generally believed that the transition occurs via nucleation and growth. This occurs because the Higgs mechanism causes the Higgs field to lose all degrees of freedom apart from the modulus $|\Psi|$, and it is this quantity which appears in the effective potential, and which varies across a

4. COSMOLOGICAL PHASE TRANSITIONS

In this section, I will present some remarks and observations on several issues of interest to cosmologists. These arose during discussions at the workshop, whilst presenting the procedures and insights obtained from simulating string formation, evolution and the dynamics of local gauge theories in condensed matter, and from inspection of computer animations of phase transition kinetics with non-scalar order parameters.

4.1. Violations of scaling

Although scaling is observed generically in the presence of a conservation law, the nonconserved order parameter dynamics only leads to scaling solutions for symmetric or critical quenches. Thus, if the order parameter obeys the TDGL with the coarse-grained free energy of eq. 11, then a symmetric or critical quench is one where the spatial average $\langle \Psi(\mathbf{x},0) \rangle = 0$ initially. However, it is not obvious that in a cosmological context, the ordering field will always satisfy this condition. For example, in a GUT transition at which strings are formed, earlier symmetry breaking transitions may create a bias in the Higgs field. In the toy-model case of the three dimensional XY model, simulations[2] confirm the prediction from mean field theory[25] that scaling is violated, with the initial string distribution rapidly breaking up into small loops. The total length of string $\ell(t)$ a time t after the quench is given by

$$\ell(t) \sim t^{-1} \exp(-b^2 t^{3/2}) \tag{16}$$

where the bias is given by $b^2 = |\langle \Psi(\mathbf{x},0) \rangle|^2$. For times less than the crossover time $t_c \sim b^{-4/3}$, the dynamics of the string network resembles that of the critical quench, but for longer times, it becomes readily apparent that there are two length scales present: the constant mean size of small loops, and the increasing mean distance between them. The success of the analytic mean field theory is actually typical. Although analytic approximations on the TDGL itself have not been successful so far, transforming the equation into an effective equation for the defects has proven to be the appropriate starting point for semi-quantitatively accurate predictions[26].

4.2. The Kibble mechanism in the GUT transition

It comes as a shock to most condensed matter physicists to learn that cosmologists have a name for the mechanism in which topological defects are formed during the quench through a phase transition. (To my knowledge, it has only occurred to one condensed matter physicist that topological defects would *not* be created during a rapid quench!) During a putative phase transition in a grand unified theory (GUT), string-like topological defects can be formed, which, many expansion times later, have been proposed to act as seeds for galaxy formation. Although the initial string density does not turn out to be important for existing models of structure formation, because the string network is believed to attain a scaling solution at long times, some attention has been given to the question of determining this quantity.

By the term "initial string density" is meant the length of string per unit volume "just after the quench". This only has a unique meaning if the quench can be considered to be instantaneous: otherwise, the order parameter configuration is determined by the history of the system, and there is no unique value for this quantity.

domain wall. There are well-known problems associated with the computation of the effective potential in this context, and the nature of the transition is currently unclear.

The main remark that I would like to make here concerns the dynamics of the phase transition. In general, it is not correct to use the derivative of $\tilde{F}\{\Psi\}$ as the driving force in a TDGL for Ψ. The reason is that the dynamics of Ψ depends upon the dynamics of the gauge field, and at least in the case of superconductors, the relevant time scales for relaxation of both Ψ and $\mathbf{A}$ are of the same order. The correct procedure is to coarse-grain both the microscopic order parameter and gauge field, and then to solve the coupled dynamical equations for both Ψ and $\mathbf{A}$. In the case of the superconductor transition, this has been done in ref. [4] with the result that growth does not proceed by nucleation and growth of bubbles. Elder and myself [28] have repeated the calculation using the effective potential $\tilde{F}$, in which the gauge field has been completely integrated out, and checked that the results are quite different from those obtained using the full dynamical equations for both the order parameter and gauge field. A similar result was obtained by Ye and Brandenberger[29] in their numerical simulations of the Abelian Higgs model, where no evidence for domain wall formation was observed. If this conclusion is supported by further investigation, then there could be important ramifications for proposed mechanisms of electroweak baryogenesis.

In conclusion, it seems that the time is ripe for a systematic study of phase transition kinetics and the dynamics of fields, not only in the context of condensed matter, but also in other areas where the space-time behaviour of fields is of interest. These include cosmology and perhaps the physics of the quark-gluon plasma, soon to be probed by RHIC.

5. ACKNOWLEDGEMENTS

The calculations of phase ordering referred to in the text were performed in collaboration with M. Mondello, F. Liu, M. Zapotocky and P. Goldbart: it is a pleasure to acknowledge their contributions. I thank Ken Elder for many discussions on phase ordering in superconductors, Yoshi Oono for his collaboration and many discussions, Alan McKane for discussions on field theory, Andy Albrecht, Rob Brandenberger, Anne Davis, Tom Kibble, Paul Shellard, Neil Turok, Alexander Vilenkin and Erick Weinberg for numerous discussions on cosmology, and the organisers of the workshop for successfully getting condensed matter physicists and cosmologists to talk to each other. Finally, I acknowledge the generous hospitality of the Isaac Newton Institute, Cambridge. This work was supported by National Science Foundation grant NSF-DMR-93-14938.

REFERENCES

[1] M. Mondello and N.D. Goldenfeld, *Phys. Rev. A* **42**, 5865 (1990).
[2] M. Mondello and N.D. Goldenfeld, *Phys. Rev. A* **45**, 657 (1992).
[3] M. Mondello and N.D. Goldenfeld, *Phys. Rev. E* **47**, 2384 (1993); see also, M. Siegert and M. Rao, *Phys. Rev. Lett.* **70**, 1956 (1993).
[4] F. Liu, M. Mondello and N.D. Goldenfeld, *Phys. Rev. Lett.* **66**, 3071 (1991); H. Frahm, S. Ullah and A. Dorsey, *Phys. Rev. Lett.* **66**, 3067 (1991).
[5] F. Liu, *Phys. Rev. E* **48**, 2422 (1993).
[6] M. Zapotocky, P. Goldbart and N.D. Goldenfeld, *Phys. Rev. E* (in press).

[7] H. Toyoki, *Phys. Rev. A* **42**, 911 (1990); *J. Phys. Soc. Jpn.* **60**, 1433 (1991); *ibid.* **60**, 1153 (1991).
[8] A.J. Bray, *Adv. Phys.* (in press).
[9] For reviews on phase ordering see: J.D. Gunton, M. San Miguel and P.S. Sahni in *Phase Transitions and Critical Phenomena*, vol. 8, eds. C. Domb and J.L. Lebowitz (Academic, New York, 1983), p. 267; K. Binder, *Rep. Prog. Phys.* **50**, 783 (1987); H. Furukawa, *Adv. Phys.* **34**, 703 (1985).
[10] A. J. Bray, *Phys. Rev. B* **41**, 6724 (1990).
[11] H. Tomita, *Prog. Theor. Phys.* **75**, 482 (1986).
[12] C. Yeung, *Phys. Rev. Lett.* **61**, 1135 (1988); H. Furukawa, *Phys. Rev. B* **40**, 2341 (1989); H. Furukawa, *J. Phys. Soc. Jpn.* **58**, 216 (1989).
[13] G. Porod, in *Small Angle X-ray Scattering*, edited by O. Glatter and L. Kratky (Academic, New York, 1983).
[14] A. Coniglio, P. Ruggiero and M. Zannetti, Scaling and Crossover in the large-N model for growth kinetics, cond-mat/9405003.
[15] A. Shinozaki and Y. Oono, *Phys. Rev. E* **48**, 2622 (1993).
[16] See (*e.g.*) the discussion by A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-particle Systems* (McGraw-Hill, New York, 1971).
[17] M. Alford and J. March-Russell, *Nucl. Phys. B* **417**, 527 (1994).
[18] For a review, see N.R. Werthamer, in *Superconductivity*, edited by R.D. Parks (Marcel Dekker, New York, 1969), vol. 1, p. 321.
[19] Y. Enomoto, *Mod. Phys. Lett. B* **5**, 1639 (1991).
[20] H. Toyoki, in *Dynamics of Ordering Processes in Condensed Matter*, ed. S. Komura, H. Furukawa (Plenum, New York, 1988).
[21] H. Pleiner, *Phys. Rev. A* **37**, 3986 (1988); P. Cladis *et al.*, *Phys. Rev. Lett.* **58**, 222 (1987); A. Pargellis *et al.*, *Phys. Rev. A* **46**, 7765 (1992); B. Yurke *et al.*, *Phys. Rev. E* **47**, 1525 (1993).
[22] Y. Oono and S. Puri, *Phys. Rev. Lett.* **58**, 836 (1987); *Phys. Rev. A* **38**, 434 (1988); S. Puri and Y. Oono, *ibid.* **38**, 1542 (1988).
[23] H. Tomita, *Prog. Theor. Phys.* **85**, 47 (1991).
[24] See L.-Y. Chen, N.D. Goldenfeld and Y. Oono, *Phys. Rev. Lett.* **73**, 1311 (1994) and references therein.
[25] H. Toyoki and K. Honda, *Prog. Theor. Phys.* **78**, 237 (1987).
[26] T. Ohta, D. Jasnow and K. Kawasaki, *Phys. Rev. Lett.* **49**, 1223 (1982).
[27] B.I. Halperin, T.C. Lubensky and S.-K. Ma, *Phys. Rev. Lett.* **32**, 292 (1974).
[28] K. Elder and N.D. Goldenfeld, unpublished.
[29] J. Ye and R. Brandenberger, *Nucl. Phys. B* **346**, 149 (1990).