Does fully developed turbulence exist? Reynolds number independence versus asymptotic covariance

G. I. Barenblatt[a)]
Department of Theoretical and Applied Mechanics, University of Illinois at Urbana—Champaign,
104 S. Wright St., Urbana, Illinois 61801 and Department of Physics and Beckman Institute,
University of Illinois at Urbana—Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080

Nigel Goldenfeld
Department of Physics and Beckman Institute, University of Illinois at Urbana—Champaign,
1110 West Green Street, Urbana, Illinois 61801-3080

By analogy with recent arguments concerning the mean velocity profile of wall-bounded turbulent shear flows, we suggest that there may exist corrections to the $\frac{2}{3}$ law of Kolmogorov, which are proportional to $(\ln \mathrm{Re})^{-1}$ at large Re. Such corrections to K41 are the only ones permitted if one insists that the functional form of statistical averages at large Re be invariant under a natural redefinition of Re. The family of curves of the observed longitudinal structure function $D_{LL}(r,\mathrm{Re})$ for different values of Re is bounded by an envelope. In one generic scenario, close to the envelope, $D_{LL}(r,\mathrm{Re})$ is of the form assumed by Kolmogorov, with corrections of $O[(\ln \mathrm{Re})^{-2}]$. In an alternative generic scenario, both the Kolmogorov constant C_K and corrections to Kolmogorov's linear relation for the third-order structure function $D_{LLL}(r)$ are proportional to $(\ln \mathrm{Re})^{-1}$. Recent experimental data of Praskovsky and Oncley appear to show a definite dependence of C_K on Re, which, if confirmed, would be consistent with the arguments given here. © *1995 American Institute of Physics.*

I. INTRODUCTION

The term *fully developed turbulence* traditionally refers to a unique state of turbulent behavior believed to occur for sufficiently large but finite Reynolds number (Re). This state is characterized by local isotropy and homogeneity and associated universal behavior of statistical properties, such as moments of the longitudinal velocity difference $v_r \equiv [\mathbf{v}(\mathbf{x}+\mathbf{r}) -\mathbf{v}(\mathbf{x})]\cdot\mathbf{r}/|\mathbf{r}|$. The first theoretical description along these lines was given by Kolmogorov and Obukhov in 1941 (referred to as K41).[1] The assumption that such a limiting state exists, and may be found at large but finite Re, is nontrivial, and in our view, has not properly been established experimentally. The purpose of this note is to investigate how a breakdown of this assumption would be manifested.

The mathematical expression of the assumption of a limiting state of fully developed turbulence is that statistical averages of the flow exhibit complete similarity[2] with respect to Re. To explain this statement, let us consider the second-order longitudinal structure function $D_{LL}(r)\equiv\langle(v_r)^2\rangle$. In K41, $D_{LL}(r)$ has the form

$$D_{LL}(r) = C_K(\bar{\epsilon}r)^{2/3}, \qquad (1)$$

where r lies in the inertial range and $\bar{\epsilon}$ is the mean rate of energy dissipation per unit mass. Kolmogorov's form for $D_{LL}(r)$ is based upon both dimensional considerations, and assumptions about limiting behavior. Dimensional analysis shows that the form of $D_{LL}(r)$ must be given by

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3}F(\mathrm{Re},r/L), \qquad (2)$$

where $F(x,y)$ is a universal function to be determined, L is the external or integral scale and r is always considered to lie in the inertial range. Kolmogorov assumed that in the limits $x\rightarrow\infty$ and $y\rightarrow0$, the function $F(x,y)$ simply takes the constant value C_K. In other words, there is *complete similarity* with respect to the variables Re and r/L.

The existence of the limit of $F(x,y)$ as $y\rightarrow0$ has been questioned[3-5] due to intermittency—fluctuations of the energy dissipation rate about its mean value $\bar{\epsilon}$. *Incomplete similarity* in the variable r/L would require the nonexistence of a finite and nonzero limit of $F(x,y)$ as $y\rightarrow0$, and leads in the simplest case to the form

$$D_{LL}(r) = C_K(\bar{\epsilon}r)^{2/3}\left(\frac{r}{L}\right)^\alpha, \qquad (3)$$

where α is the so-called intermittency exponent, believed to be small and non-negative.

In the present paper, we argue that there may be an alternative way, which we term *asymptotic covariance*, in which a lack of complete similarity can occur. We try to use physical arguments to constrain the mathematical form that this might take. Asymptotic covariance in Re would imply that there is no unique limiting state of fully developed turbulence. Instead, the manner in which statistical averages evolve with Re for large Re is governed by a functional form that in the simplest case is universal.

Precisely the same set of arguments can be made for a seemingly different, but related problem: the mean velocity profile in a wall-bounded turbulent shear flow. There is a well-known (and probably superficial) analogy[6] between the boundary layers and universal scaling regimes of both the spatial structure of wall-bounded turbulent shear flows and the local structure of turbulence. For example, consider tur-

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[a)]Permanent address: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver St., Cambridge CB3 9EW, United Kingdom.

bulent flow in a pipe. The viscous wall layer is analogous to the dissipative range in fully developed turbulence; the velocity profile (conventionally described by the universal von Kármán–Prandtl logarithmic law) outside the viscous wall layer, but on scales much smaller than the pipe radius is analogous to the inertial range (conventionally described K41); and the nonuniversal finite-size effects on the flow associated with scales of order the pipe radius are analogous to the nonuniversal behavior of fully developed turbulence at the integral scale. This problem was already considered in detail[7] and it was shown that the existing data do not exclude the possibility there of asymptotic covariance in Re.

This paper is organized as follows. In Sec. II, we review the analysis of the wall-bounded shear flow. In particular, we propose a principle of *asymptotic covariance*, in which we insist that the functional form of statistical averages at large Re be invariant under redefinition of Re. This implies that a form of incomplete similarity must occur in terms of the variable ln Re. In Sec. III, we consider the analogous arguments for the local structure of fully developed turbulence. We conclude in Sec. IV with a brief discussion of experimental data and some final comments.

II. WALL-BOUNDED TURBULENT SHEAR FLOWS

We consider a wall-bounded shear flow that is statistically steady and homogeneous in the longitudinal direction. Its properties vary only in the lateral direction, perpendicular to the wall. A classic example, which we shall always have in mind, is the flow in a pipe far from the entrance and outlet.

A. Mean velocity profile

von Kármán[8] and Prandtl[9] obtained the law for the variation of the mean longitudinal velocity $\bar{u}$ in an intermediate region of the turbulent shear flow, outside a small "viscous" sublayer near the wall. Within this sublayer, the stress due to molecular momentum transfer is comparable in magnitude with that due to turbulent momentum transfer by vortices. The universal, Reynolds number independent von Kármán–Prandtl logarithmic law has the form

$$\phi = \frac{1}{\kappa} \ln \eta + \tilde{C}, \tag{4}$$

where

$$\phi \equiv \bar{u}/u_*, \quad \eta \equiv u_* y/\nu, \quad u_* \equiv (\tau/\rho)^{1/2}, \tag{5}$$

y is the distance from the wall, and ρ and ν are the fluid density and kinematic viscosity, respectively. The constants κ (the von Kármán constant) and $\tilde{C}$ are universal according to the logic of the derivation. The logarithmic law follows from a strong assumption of complete similarity, namely that in the intermediate region the contribution of the molecular viscosity and the external length scale (e.g., the diameter of the pipe) could be completely neglected.

It was shown in Ref. 7 that this assumption is questionable, and an alternative relationship was proposed, corresponding to incomplete similarity:

$$\phi = C \eta^\alpha, \tag{6}$$

where the coefficients C and α may be expressed in terms of the small parameter,

$$\epsilon \equiv \frac{1}{\ln \text{Re}}, \tag{7}$$

by the expansions

$$\alpha = \frac{3}{2}\epsilon + O(\epsilon^2), \quad C = \frac{1}{\sqrt{3}\epsilon} + \frac{5}{2} + O(\epsilon). \tag{8}$$

It is important that the power α in (6) depends upon the Reynolds number for two reasons. First, it is well known that a Re-independent power law form is inconsistent with the data.[10]

Second, Eyink[11] has examined whether or not a Re-independent power law is inconsistent with the rigorous bound on the energy dissipation rate given by Doering and Constantin,[12] (making the plausible assumption that their bound can be carried over to the pipe geometry), and has shown that the form given in (8) is consistent with the bound. Eyink's argument is as follows: the average dissipation per unit mass $\bar{\epsilon}$ is

$$\bar{\epsilon} = \frac{1}{L} \int_0^L \bar{\epsilon}'(y) dy, \quad \bar{\epsilon}'(y) = u_*^2 \frac{\partial \bar{u}}{\partial y}, \tag{9}$$

which for the von Kármán–Prandtl law (4) can be estimated to be

$$\bar{\epsilon} = \frac{U^3}{L} \frac{1}{\kappa (\ln \text{Re})^2}, \quad \text{Re} \to \infty. \tag{10}$$

For the velocity profile (6), however, it can be shown that

$$\bar{\epsilon}'(y) \sim \left(\frac{L}{y}\right)^{1-\alpha} [2^\alpha \alpha (\alpha+1) \\ \times (\alpha+2)]^{(3+\alpha)/(1+\alpha)} \text{Re}^{-2\alpha/(1+\alpha)}. \tag{11}$$

If α were Re independent, the dissipation in the intermediate region would vanish sharply as Re→∞. However, a more physically plausible alternative is that there is only a weak dependence of the dissipation on Re for Re→∞:

$$\text{Re}^\alpha \sim \text{const}, \quad \text{i.e.,} \quad \alpha \sim \frac{\text{const}}{\ln \text{Re}}. \tag{12}$$

The corresponding result for the dissipation is that

$$\bar{\epsilon} \sim \frac{U^3}{L} \frac{1}{(\ln \text{Re})^2}, \quad \text{Re} \to \infty, \tag{13}$$

which is very similar to the estimate for the von Kármán–Prandtl law.

It seems that the scaling law (6) manifests a lack of universality, by virtue of its dependence on Re. However, this is not so, in the following sense. Instead of the traditional universal straight line in the ϕ−ln η plane corresponding to (4), there is a one-parameter family of curves (6) occupying a certain region of the plane. This region is nevertheless universal, in the sense that it is bounded by the envelope of the family, which is a universal curve. The equation for the envelope is obtained by eliminating ln Re from

(6), written in the form $\phi = F(\ln \eta, \ln \text{Re})$, and the tangency condition $\partial F / \partial \ln \text{Re} = 0$, and is found to have the universal form

$$\phi = 5\left(\sigma^{-1} + \frac{1}{2}\right)\eta^{\sigma\sqrt{3}/10}, \quad \sigma \equiv \left(1 + \frac{20}{\sqrt{3}\,\ln \eta}\right)^{1/2} - 1.$$
(14)

This envelope is, in fact, close to (4), even for moderate values of $\ln \eta$ and Re, if $\kappa \sim 0.4$ and $\tilde{C} = 5.1$. For presently unachievably large values of Re and $\ln \eta$, the universal envelope (14) assumes the form

$$\phi = \frac{\sqrt{3}e}{2}\ln \eta + \frac{5e}{2},$$
(15)

where the coefficient $2/\sqrt{3}e \approx 0.424$ is close to the generally accepted value of the von Kármán constant κ, but $5e/2 \approx 6.79$ is larger than the generally accepted value of $\tilde{C} \approx 5.1-5.5$.

Thus, even when there is a lack of similarity, it seems that for $\ln \text{Re} \gg 1$ in the intermediate region of the shear flow, the state is described by the relation $\phi = F(\ln \eta, \ln \text{Re})$, universal in the sense that the same function F applies to all turbulent shear flows.

B. Asymptotic covariance

It is instructive that the universal relation (6)–(8) contains Re only through the dependence on $\ln \text{Re}$. In fact, this is inevitable, and a consequence of what we propose to call *asymptotic covariance*. Asymptotic covariance provides a general constraint on the way in which a lack of complete similarity in Re may be exhibited in a turbulent flow. Let us assume that we consider a simple turbulent flow, such as that in a pipe, and that a putative state of isotropic, homogeneous turbulence is present on some scale $r \ll L$, where L is taken to be the diameter of the pipe. The *nature* of the turbulent state should be insensitive to small changes in the cross-sectional average input and output flow rate U or the diameter L. Thus, we require that at sufficiently large Re, the *functional form* of statistical averages, such as $D_{LL}(r)$ or ϕ, should not be influenced by a redefinition of Re, such that

$$\text{Re} \to \text{Re}' \equiv Z\,\text{Re} = \text{Re} + \delta/\text{Re}, \quad \delta/\text{Re}/\text{Re} \to 0. \quad (16)$$

For example, at sufficiently large Re, we could use for L the radius of the pipe instead of the diameter, or for U the maximum velocity instead of the average one etc., without changing the functional form of $D_{LL}(r)$ or ϕ. Let us suppose that we consider such a statistical average, whose dependence on Re (and η in the case of the shear flow considered in this section) is through its dependence on a function $F[\ln \eta, \psi(\text{Re})]$ having for definiteness a uniformly bounded first derivative, and with ψ an unboundedly growing function of its argument, to be determined by the following considerations.

For any $Z > 0$,

$$\psi(\text{Re}) = \psi(\text{Re}_0) + \int_{\text{Re}_0}^{\text{Re}} \frac{d\psi(\text{Re}')}{d\,\text{Re}'}\,d\,\text{Re}',$$
(17)

$$\psi(Z\,\text{Re}) = \psi(Z\,\text{Re}_0) + \int_{\text{Re}_0}^{\text{Re}} \frac{d\psi(Z\,\text{Re}')}{d\,\text{Re}'}\,d\,\text{Re}'.$$

Here, Re_0 is some reference value of Re. The first term on the right-hand sides of (17) at large Re is small in comparison with the second term, because ψ is unbounded. However, $d\psi(Z\,\text{Re})/d\,\text{Re} = Z\psi'(Z\,\text{Re})$, where $'$ denotes differentiation with respect to the argument. Asymptotic covariance is equivalent to the statement that

$$\psi'(\text{Re}) = \psi'(Z\,\text{Re}). \quad (18)$$

Thus, the right-hand side of (18) is simply $Z\psi'(Z\,\text{Re})$, and

$$\text{Re}\,\psi'(\text{Re}) = Z\,\text{Re}\,\psi'(Z\,\text{Re}), \quad (19)$$

giving $\psi(\text{Re}) \propto \ln \text{Re}$.

Let us now consider the question of whether one could use the Taylor microscale Reynolds number Re_λ, which is often assumed to vary approximately as $\sqrt{\text{Re}}$. From the point of view of advanced similarity methods and dimensional analysis, the important point about Re is that it represents a characterization of the system that can be made *a priori*. That is, it is not an emergent property of the flow (i.e., depending upon the *solution* of the equation of motion), but a property of the *constraints* or *boundary conditions* placed upon the flow. On the other hand, Re_λ represents the response of the flow, and, in general, for arbitrary Re, will not necessarily have a unique, universal dependence on Re.[13] However, for an asymptotically covariant theory, at very large Re, the variation of $\ln \text{Re}_\lambda$ with $\ln \text{Re}$ may be written in the form

$$\frac{d\,\ln \text{Re}_\lambda}{d\,\ln \text{Re}} = a_0 + \frac{a_1}{\ln \text{Re}} + O\left(\frac{1}{(\ln \text{Re})^2}\right), \quad (20)$$

where a_0, a_1, etc. are constants, so that

$$\text{Re}_\lambda \sim \text{Re}^{a_0}(\ln \text{Re})^{a_1}, \quad (21)$$

showing that there is a unique power law relationship between Re_λ and Re.

III. LOCAL STRUCTURE OF FULLY DEVELOPED TURBULENCE

As explained in the Introduction, Kolmogorov's form (1) for

$$D_{LL}(r) = (\bar{\epsilon}r)^{2/3}F(\text{Re}, r/L), \quad (22)$$

is based upon both dimensional considerations and assumptions about limiting behavior. Let us now examine the question of the lack of complete similarity of (2), based upon considerations analogous to those used above for wall-bounded turbulent shear flows.

We will assume that the spatial fluctuations in the energy flux diminish at higher Re, because the flow configurations or processes that correspond to these fluctuations become increasingly dense throughout the flow. In this picture, the scaling assumed in K41 becomes more and more accurate at very high Re. We will refer to this assumption as *asymptotic Kolmogorov scaling*.[14]

The behavior of $F(x, y)$ as $x \to \infty$ is the central issue on which we focus. The manner in which F could fail to attain a finite limit is constrained by asymptotic covariance, which implies that

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$$D_{LL}(r)=(\bar\epsilon r)^{2/3}G(\ln \mathrm{Re},r/L), \tag{23}$$

for some function G to be determined, and intermittency, i.e. incomplete similarity of $G(x,y)$ in the limit $y\to 0$. Conventionally, it is assumed that the manner of violation of K41 is that $G(x,y)\sim y^\alpha$ as $y\to 0$, where α is an intermittency correction, assumed to be Re independent and estimated experimentally to be positive. In contrast, we wish to investigate the consequences of assuming that as $x\to\infty$ and $y\to 0$,

$$D_{LL}(r)=(\bar\epsilon r)^{2/3}A(\ln \mathrm{Re})\left(\frac{r}{L}\right)^{\alpha(\ln \mathrm{Re})}, \tag{24}$$

where the prefactor A and intermittency exponent α depend on $\ln \mathrm{Re}$. In particular, the assumption of asymptotic covariance implies that we can write

$$\alpha=\alpha_0+\frac{\alpha_1}{\ln \mathrm{Re}}+O[(\ln \mathrm{Re})^{-2}]; \tag{25}$$

further assuming asymptotic Kolmogorov scaling implies that $\alpha_0=0$. Such a form is consistent with the experimental results of Castaing et al.[15] As before, instead of a universal straight line in the $D_{LL}(r)/(\bar\epsilon r)^{2/3}-(r/L)$ plane, we obtain a one parameter family of curves, occupying a certain portion of the plane. The boundary of this family is a universal, Re-independent curve, which satisfies both (24) and the condition $\partial D_{LL}(r)/\partial\mathrm{Re}=0$, i.e.

$$\frac{dA}{d(\ln \mathrm{Re})}=\dot A(\ln \mathrm{Re})\frac{\alpha_1\ln(r/L)}{(\ln \mathrm{Re})^2}. \tag{26}$$

Empirically it is found that the intermittency correction α is positive, so that $\alpha_1>0$. However, in the inertial range, $r\ll L$, so that $dA/d(\ln \mathrm{Re})<0$. Thus, we may expand A in the small parameter $\epsilon\equiv 1/\ln \mathrm{Re}$,

$$A(\ln \mathrm{Re})=A_0+A_1\epsilon+O(\epsilon^{-2}), \tag{27}$$

where A_0, A_1 are non-negative constants. There are two cases to consider: (a) $A_0\neq 0$, and (b) $A_0=0$. We will briefly discuss the experimental estimates[16] of C_K in the following section, but for now, we remark that both the scatter and the recent report[17] of a systematic dependence on Re encourage us to consider not only the conventional case (a) but also case (b).

In case (a), both intermittency corrections to the exponent α and to the Kolmogorov constant A_0 vanish logarithmically as $\ln \mathrm{Re}\to\infty$. More precisely, the relation (26) gives

$$\frac{\alpha_1\ln(r/L)}{\ln \mathrm{Re}}=-\frac{A_1}{\ln \mathrm{Re}}\frac{1}{\{A_0+A_1/\ln \mathrm{Re}+O[(\ln \mathrm{Re})^{-2}]\}}, \tag{28}$$

so that the envelope is given by

$$D_{LL}^{\mathrm{env}}(r)=(\bar\epsilon r)^{2/3}A_0\{1+O[\ln(r/L)]\}. \tag{29}$$

Note the correction term in this formula: the cancellations that occur to first order in ϵ make the deviations from K41 very difficult to detect. Indeed, near the envelope, individual plots of $D_{LL}(r)$ at fixed but large Re will exhibit corrections to K41 only of order $O(\epsilon^2)$.

In case (b), the results are more dramatic. The envelope condition (26) becomes

$$\frac{dA}{d(\ln \mathrm{Re})}=-\frac{A}{\ln \mathrm{Re}} \tag{30}$$

so that we obtain the condition

$$1=-\frac{\alpha_1\ln(r/L)}{\ln \mathrm{Re}}. \tag{31}$$

The universal boundary in the $D_{LL}(r)/(\bar\epsilon r)^{2/3}-(r/L)$ plane is then represented by the curve

$$\frac{D_{LL}(r)}{(\bar\epsilon r)^{2/3}}=-\frac{A_1}{e\,\alpha_1\ln(r/L)}. \tag{32}$$

The equation of a member of the family of curves at fixed Re near the envelope is

$$D_{LL}(r)=(\bar\epsilon r)^{2/3}\frac{A_1}{e\ln \mathrm{Re}}, \tag{33}$$

which is K41, but with a Kolmogorov constant $C_K=A_1/e\ln \mathrm{Re}_\lambda$.

The phenomenological considerations given above also have implications for higher moments. For example, consider the third moment $D_{LLL}(r)\equiv\langle(v_r)^3\rangle$, which is related to $D_{LL}(r)$ by Kolmogorov's relation,

$$D_{LLL}(r)=-\frac{4}{5}\bar\epsilon r+6\nu\frac{dD_{LL}(r)}{dr}. \tag{34}$$

Using the form (24) for $D_{LL}(r)$, and assuming case (b) above, we find that for large $\ln \mathrm{Re}$,

$$D_{LLL}(r)=-\frac{4}{5}\bar\epsilon r\left(1-\frac{5A_1}{\ln \mathrm{Re}}\left(\frac{\eta}{r}\right)^{4/3}\right.$$
$$\left.\times e^{-3\alpha_1/4}+O[(\ln \mathrm{Re})^{-2}]\right). \tag{35}$$

The correction to Kolmogorov's "$\frac{4}{5}$ law" is Reynolds number dependent, in contrast to the situation in K41.

IV. DISCUSSION

Sreenivasan[16] recently surveyed the available data on C_K, for different flows and Re, and came to the conclusion that there is no Reynolds number dependence for the data taken as a whole. However, there is considerable scatter in the data, and as noted by Sreenivasan, a controlled series of measurements in a single flow geometry for a wide range of values of Re has yet to be performed.

To our knowledge, the only measurements that explore the systematics of the possible Reynolds number dependence of C_K are those of Praskovsky and Oncley,[17] in which a weak dependence of C_K on Re is reported. These authors attempted to fit this dependence with a power law, but, as shown in Fig. 1, their data are consistent with the logarithmic forms proposed here. Figure 1 shows the data, together with two best fits to the form given by (27), for the two cases $A_0\neq 0$ and $A_0=0$. The former case, with a nonzero limit of C_K as $\mathrm{Re}\to\infty$, may be a slightly better fit than the latter. One measure of this is shown in the insets of Fig. 1: Inset (a) shows the scatter when $C_K\ln \mathrm{Re}_\lambda$ is plotted against Re_λ, while inset (b) shows the scatter when $\Delta C_K\ln \mathrm{Re}_\lambda$ is plotted
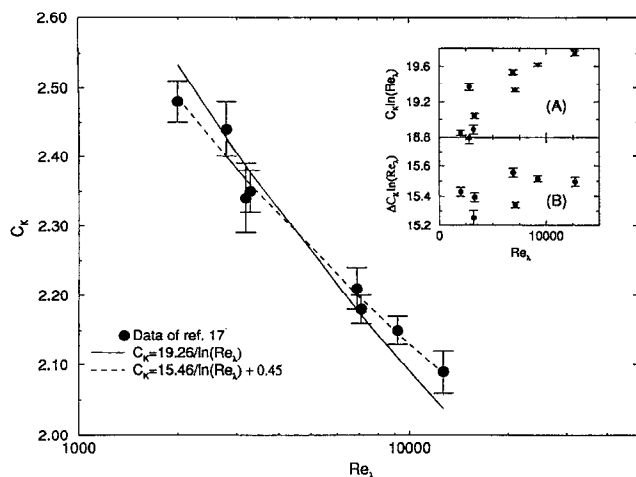
FIG. 1. The Kolmogorov constant, as measured by Praskovsky and Oncley,[17] plotted on a logarithmic scale against Taylor microscale Reynolds number. Also shown are best fits to the functional form given by (27), for the two cases $A_0 \neq 0$ and $A_0 = 0$. Inset (a) shows the scatter when $C_K \ln \mathrm{Re}_\lambda$ is plotted against Re_λ. Inset (b) shows the scatter when $\Delta C_K \ln \mathrm{Re}_\lambda$ is plotted against Re_λ, where $\Delta C_K \equiv C_K - 0.45$.

against Re_λ, where $\Delta C_K \equiv C_K - 0.45$. A slight upward trend with increasing Re_λ may be discernable in (a), whereas the scatter in (b) seems to be more uniform. Clearly, these data (with only eight points) are not sufficient to draw any strong conclusions; nevertheless, they are not inconsistent with either of the two possibilities $A_0 \neq 0$ or $A_0 = 0$ in (27).

Therefore, it would be of considerable fundamental value to obtain a confirmation (or otherwise) of the results of Praskovsky and Oncley, with greater precision. In particular, evidence that $A_0 = 0$ would indicate that fully developed turbulence is not a unique, Reynolds number independent state, approached at large enough Reynolds number, but instead is a state with universal Reynolds number and external scale dependence, identical for different flows, but with no attainable limit.

In conclusion, we have made two main points in this paper. First, we have suggested that there may not be complete similarity in Re—in particular, we have suggested that the lack of complete similarity is exhibited as asymptotic covariance—and we have detailed some of the consequences of this. In particular, a universal scaling should be approached at large Re. Second, the universal scaling may be that of K41, indicating the existence of a unique fully developed turbulent state, or may have no asymptotic limit, indicating the nonexistence of such a simple state.

ACKNOWLEDGMENTS

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