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Renormalization-group theory for the propagation of a turbulent burst

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We consider the propagation of a plane front separating a turbulent region of fluid from a quiescent region. Initially, the turbulent-energy distribution as a function of z, the displacement normal to the front, is assumed to be localized, and after a time t, general renormalization-group arguments show that there is a similarity solution of the form $q(z,t)\sim t^{-(2/3+2\tilde{\alpha})}f(zt^{-(2/3+\beta)},\ \epsilon)$, where $\tilde{\alpha}$ and β are ϵ-dependent anomalous dimensions, satisfying the scaling law $\tilde{\alpha}+\beta=0$ and ϵ is a measure of the dissipation. Using perturbation theory, we calculate values of $\tilde{\alpha}$ and β to $O(\epsilon)$, which are in good agreement with numerical calculations, and we explicitly verify the above scaling law and find the form of the scaling function f.

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I. INTRODUCTION

Dimensional analysis is a powerful way to study those physical problems in which one is interested in the long-time or large-scale asymptotic behavior as one of the dimensionless quantities in the problem Π_0 tends to zero or infinity. If the quantity of interest Π is expressed in the dimensionless form $\Pi=f(\Pi_0,\Pi_1,\ldots,\Pi_n)$, as a function f of the dimensionless quantities in the problem $\Pi_0,\Pi_1,\ldots,\Pi_n$, then it is often assumed that the limit as $\Pi_0\to 0$ is well defined:

$$\lim_{\Pi_0\to 0}\Pi=f(0,\Pi_1,\Pi_2,\ldots,\Pi_n)\ . \tag{1.1}$$

However, there are a number of situations, as Barenblatt has pointed out [1], where simple dimensional analysis often fails: the function f is not well behaved in the limit $\Pi_0\to 0$. In the case of intermediate asymptotics of the second kind, the correct behavior is given by

$$\lim_{\Pi_0\to 0}\frac{\Pi}{\Pi_0^{\alpha}}=\lim_{\Pi_0\to 0}\Pi_0^{-\alpha}f\left[\Pi_0,\frac{\Pi_1}{\Pi_0^{\alpha_1}},\ldots,\frac{\Pi_n}{\Pi_0^{\alpha_n}}\right]\ . \tag{1.2}$$

with an appropriate set of real exponents $\alpha,\alpha_1,\ldots,\alpha_n$ which cannot be determined by dimensional analysis alone, but which may be determined only by solving the full problem itself. Interesting physical examples *without any statistical aspect* occur in shock-wave propagation [1], in fluid flow in porous media [1], and possibly in velocity selection in dendritic growth [2]. Other well-known examples [3] which do have a statistical aspect are critical phenomena [4] and spinodal decomposition [5].

Recently, it was shown [6–8] that the exponents $\alpha,\alpha_1,\ldots,\alpha_n$ are just the anomalous dimensions of the renormalization group (RG) in field theory, and can be computed even in those cases without statistical phenomena or noise, by using the scheme of perturbative Gell-Mann–Low RG [6,7,9] or the fixed-point formulation of Wilson [3,8,10]. This RG approach has been successfully applied to several diverse problems, such as the Barenblatt equation, describing the pressure of a fluid during its filtration through an elastoplastic porous medium [1,6–8], the modified porous-medium equation governing a number of situations, such as the height of a groundwater mound during gravity-driven flow in porous media and the propagation of strong thermal waves following an intensive explosion [11], convection-diffusion transport with irreversible sorption [12], and two linear problems in continuum mechanics [13].

The purpose of this present paper is to extend the perturbative RG approach to the problem of the propagation of turbulence from an instantaneous plane source with a finite initial layer depth [14]. We emphasize that our application of the RG is different from that which has been proposed to model fully developed turbulence [15–17]. There, a random force term is added to the Navier-Stokes equation, and with a judiciously chosen noise correlation, Kolmogorov scaling is recovered. In contrast, our formulation does not require any *ad hoc* noise terms; whether or not our approach will be useful in studying fully developed turbulence is still an open question.

Following Barenblatt [14], we consider an infinite space filled with an incompressible and initially homogeneous fluid at rest. At the initial time $t=0$, we suppose that a plane turbulent layer of thickness $2a$ is formed instantaneously. Subsequently, the turbulent layer expands and propagates into the surrounding fluid. Since the flow is presumed to be shearless, the turbulent energy can only be dissipated into heat and decays gradually in the process of expansion.

The governing equation for the flow is the turbulent-energy-balance equation, which can be written as [18]

$$\partial_t q=\partial_z(\kappa_q\partial_z q)-E_t\ , \tag{1.3}$$

where the z axis is taken along the normal to the middle plane of the layer and the turbulent-energy distribution is assumed symmetric with respect to this plane, $q(z,t)$ is the mean turbulent energy per unit mass, κ_q is the turbulent-energy eddy diffusion coefficient, which in principle can depend on q, and E_t is the mean rate of turbulent-energy dissipation per unit mass.

According to the Kolmogorov similarity hypothesis [18,19], the local values of κ_q and E_t depend only on the mean eddy size and energy. Simple dimensional analysis then yields [14]

$$\kappa_q = l\sqrt{q} \ , \quad E_t = \frac{\epsilon q^{3/2}}{l} \ , \tag{1.4}$$

where ϵ is a constant and l is an unknown length scale characteristic of the turbulent eddy size. Due to the nonlinearity of Eq. (1.3) when combined with Eq. (1.4), for a bounded compact support initial distribution of the turbulent energy and an arbitrary bounded l, the turbulent energy propagates with a finite speed and remains bounded everywhere at later times. It is natural to assume that l is related to the actual turbulent layer half depth $h(t)$ by $l=\alpha h(t)$, where $\alpha<1$ is a constant, which cannot be determined by phenomenological considerations alone [14]. In summary, Eq. (1.3) is reduced to the following closed form:

$$\partial_t q = \partial_z[\alpha h(t)\sqrt{q}\,\partial_z q] - \frac{\epsilon q^{3/2}}{\alpha h(t)} \ , \tag{1.5}$$

with q different from zero only for $-h(t)\le z\le h(t)$. Note that $h(t)$ is *a priori* unknown and must be determined in the course of solving the equation.

In Sec. II, we use general RG arguments to predict the actual form of the long-time behavior of the turbulent energy equation with dissipation, where $\epsilon\neq0$, and to derive a scaling law satisfied by the exponents appearing in the long-time behavior. Although the problem as posed does not have a similarity solution, at long times the solution is arbitrarily close to a certain similarity solution. In Sec. III, we perform perturbative RG calculations to determine the actual values of these exponents and the form of the scaling functions.

The analysis in the present article parallels that which we have given for other initial-value problems with anomalous dimensions, and the reader unfamiliar with the technique is encouraged to peruse Refs. [7,8,11,13], which are more pedagogical than the present article.

II. GENERAL RENORMALIZATION-GROUP ANALYSIS

In this section, we consider the initial-value Cauchy problem, under the initial distribution of the turbulent energy

$$q(z,0) = \frac{Q_a}{a} u\left[\frac{z}{a}\right] \ , \quad Q_a = \int_{-a}^{a} q(z,0)dz \ ,$$
$$\int_{-1}^{1} u(x)dx = 1 \tag{2.1}$$

with $-a\le z\le a$, where $2a$ is the initial depth of the turbulent layer and Q_a is the finite bulk intensity of the instantaneous turbulence source. The solution of this problem can only depend on the governing parameters $Q_a,t,z,\alpha,\epsilon,a$ as $q=q(Q_a,t,z,a,\alpha,\epsilon)$. The turbulent-energy density per unit mass q has dimensions LT^{-2}. Choosing z and t as the two independent dimensional quantities, we obtain from dimensional analysis

$$\Pi = f(\Pi_1,\Pi_2,\Pi_3,\Pi_4) \ , \tag{2.2}$$

where

$$\Pi \equiv \frac{qt^{2/3}}{Q_a^{2/3}} \ , \quad \Pi_1 \equiv \frac{z}{Q_a^{1/3}t^{2/3}} \ , \quad \Pi_2 \equiv \frac{a}{Q_a^{1/3}t^{2/3}} \ ,$$
$$\Pi_3 \equiv \alpha, \quad \Pi_4 \equiv \epsilon \ . \tag{2.3}$$

In contrast to the problems in Refs. [7] and [11], here the turbulent energy per unit mass is not a quantity independent of z and t. The long-time aymptotic behavior of the initial-value problem may be obtained by taking the limit $\Pi_2\to0$. In the case $\epsilon=0$, it can be shown that this result is regular as $a\to0$, and a similarity solution is obtained by simply setting $\Pi_2=0$ in Eq. (2.2) [14]. However, in the case of nonzero dissipation ($\epsilon\neq0$), there is no such similarity solution of the form $\Pi=f(\Pi_1,0,\Pi_3,\Pi_4)$ corresponding to $a=0$ and $Q_a\neq0$, as shown by Barenblatt [14]. Due to energy dissipation, the total turbulent energy $\int_{-h(t)}^{h(t)}q(z,t)dz$ is not conserved so that Q_a is not observable at later times. The observed quantity Q is related to the unobservable quantity Q_a by

$$Q = Z^{-1}Q_a \ , \tag{2.4}$$

where Z is a so-called renormalization constant [20] which depends on a and ϵ and $Q=\int_{-h(t)}^{h(t)}q(z,t)dz$. The meaning of these statements is as follows. For a time $t>0$, the quantity $q(z)$ may be inferred. However, the origin of time cannot be inferred from measuring $q(z)$, so there is no unique initial condition which can give rise to the observed $q(z)$. In fact, there is a family of initial conditions, parametrized by a, which can give rise to the observed $q(z)$ for different $t>0$. The observed Q is proportional to the initial Q_a (they have the same units) and Z expresses how they are related. In particular, Eq. (2.4) is valid in the limit $a\to0$. In this limit, we will see that Q_a diverges, but Q remains fixed of course, being independent of a. We will show that the divergence of Q_a can be compensated by Z. Since Z is a dimensionless quantity, by dimensional analysis another additional arbitrary length scale μ must be introduced in the problem, and Z has the functional form $Z=Z(a/\mu,\epsilon)$. Substituting Eq. (2.4) in Eq. (2.2), therefore, we have

$$q(z,t) = \frac{(ZQ)^{2/3}}{t^{2/3}} f(\xi,\eta,\epsilon,\alpha,\sigma) \ , \tag{2.5}$$

where

$$\xi \equiv \frac{z}{(ZQ)^{1/3}t^{2/3}} \ , \quad \eta \equiv \frac{\mu}{(ZQ)^{1/3}t^{2/3}} \ , \quad \sigma \equiv \frac{a}{\mu} \ . \tag{2.6}$$

The actual solution q cannot depend on the arbitrary length scale μ because μ is not present in the original problem. Thus we obtain the renormalization-group equation

$$\mu \frac{\partial q}{\partial \mu}\bigg|_{Q_a,a,z,t,\epsilon,\alpha,\sigma} = 0 \ . \tag{2.7}$$

In the limit $a\to0$, assuming that $\partial f/\partial\sigma=0$ as $\sigma\to0$ and that the limit

$$\gamma \equiv \left. \frac{d \ln Z}{d \ln \mu} \right|_{a \to 0}$$

exists, we find that

$$\frac{\gamma}{3}\xi \frac{\partial f}{\partial \xi} + \left[1+\frac{\gamma}{3}\right]\eta \frac{\partial f}{\partial \eta} - \frac{2}{3}\gamma f = 0 \ . \qquad (2.8)$$

The assumptions above are nontrivial, and in statistical field theory amount to the assumption of renormalizability [20]. Here, we cannot justify them without further analysis; we believe that they are equivalent to assuming existence and uniqueness of the Cauchy problem. In previous work on the Barenblatt equation [21,22] these properties had been rigorously demonstrated. In the present case, the failure of these assumptions would be visible, because our renormalization procedure would not be consistent; however, nothing in our analysis leads us to doubt that the present problem is indeed renormalizable.

Using the method of characteristics, we obtain the general solution $q(z,t)$ of the form

$$q(z,t) \sim t^{-(2/3+2\tilde{\alpha})} F\left[\frac{z}{t^{2/3+\beta}};\alpha,\epsilon\right] \qquad (2.9)$$

with F a scaling function to be determined, and anomalous dimensions

$$\tilde{\alpha} = \frac{2\gamma}{9+3\gamma} \ , \quad \beta = -\frac{2\gamma}{9+3\gamma} \ . \qquad (2.10)$$

The anomalous dimensions $\tilde{\alpha}$ and β satisfy the scaling law

$$\tilde{\alpha}+\beta = 0 \ . \qquad (2.11)$$

Using the same sort of analysis, we are able to investigate the long-time behavior of the turbulent layer half depth $h(t)$:

$$h(t) = Q_a^{1/3} t^{2/3} F_h\left[\frac{a}{Q_a^{1/3} t^{2/3}};\alpha,\epsilon\right] , \qquad (2.12)$$

where F_h is another scaling function. After renormalization we have

$$h(t) = (ZQ)^{1/3} t^{2/3} F_h\left[\frac{\mu}{(ZQ)^{1/3} t^{2/3}};\sigma,\alpha,\epsilon\right] . \qquad (2.13)$$

From the RG equation

$$\mu \left. \frac{\partial h}{\partial \mu} \right|_{Q_a,a,z,t,\epsilon,\alpha,\sigma} = 0 \qquad (2.14)$$

we have

$$\frac{\gamma}{3} F_h + \left[1+\frac{\gamma}{3}\right]\eta \frac{\partial F_h}{\partial \eta} = 0 \ , \qquad (2.15)$$

which yields

$$F_h \sim \eta^{\gamma/(3+\gamma)} \ , \qquad (2.16)$$

and so

$$h(t) \sim t^{2/3+\tilde{\beta}} \ , \quad \tilde{\beta} = \beta = -\frac{2\gamma}{9+3\gamma} \ . \qquad (2.17)$$

These results are the starting assumptions used by Barenblatt [14].

III. PERTURBATIVE RENORMALIZATION-GROUP THEORY

In this section, we construct a naive perturbation theory in ϵ to investigate the actual form of the solution of the turbulent-energy-balance equation

$$\partial_t q = \partial_z[\alpha h(t)\sqrt{q}\, \partial_z q] - \frac{\epsilon q^{3/2}}{\alpha h(t)} \ . \qquad (3.1)$$

We take the initial distribution

$$q(z,0) = \frac{Q_a}{a}\frac{\xi_0^3}{36\alpha^2}\left[1-\frac{z^2}{a^2}\right]^2 \Theta(a-|z|) \ , \qquad (3.2)$$

which satisfies the bounded normalization condition (2.1), where Θ is the Heaviside step function and $\xi_0 = (135\alpha^2/4)^{1/3}$. This is in fact obtained from the similarity solution for the case $\epsilon=0$: we start from a δ function, evolved forward in time until the front position was at $z=a$, and take this distribution as the initial condition. In previous work on the porous medium equation, we have verified that none of our conclusions depend on the form of $q(z,0)$, as expected on general grounds. The form we have taken is, however, very convenient for analytic calculations. In order to perform the perturbation calculation, it is convenient to make the transformation of Eq. (3.1) to the "Hamilton-Jacobi" form, by setting $v = q^{1/2}$. The resultant equation for v is

$$\partial_t v = \alpha h(t)\left[v\frac{\partial^2 v}{\partial z^2} + 2\left[\frac{\partial v}{\partial z}\right]^2\right] - \frac{\epsilon v^2}{2\alpha h(t)} \ . \qquad (3.3)$$

We make the naive ϵ expansions of $v(z,t)$ and $h(t)$:

$$v(z,t) = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots \qquad (3.4)$$

and

$$h(t) = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \cdots \ . \qquad (3.5)$$

The zeroth-order equation has the form

$$\partial_t v_0 = \alpha h_0(t)\left[v_0\frac{\partial^2 v_0}{\partial z^2} + 2\left[\frac{\partial v_0}{\partial z}\right]^2\right] \qquad (3.6)$$

with the solution

$$v_0(z,t) = \frac{Q_a^{1/2}\xi_0^{3/2}}{6\alpha h_0(t)^{1/2}}\left[1-\frac{z^2}{h_0(t)^2}\right]\Theta(h_0(t)-|z|) \ , \qquad (3.7)$$

where $h_0(t) = (\xi_0^{3/2} Q_a^{1/2} t + a^{3/2})^{2/3}$. At large times $t \gg a^{3/2}/\xi_0^{3/2} Q_a^{1/2}$, this solution tends toward the self-similar solution

$$\lim_{t\to\infty} v_0(z,t) \sim \frac{1}{t^{1/3}} f\left[\frac{z}{t^{2/3}}\right] \quad (\epsilon=0) \qquad (3.8)$$

with $\lim_{t\to\infty} h_0(t) \sim t^{2/3}$, allowing us to read off the scaling function f from Eq. (3.7). In the following calculations, we achieve the limit $t \gg a^{3/2}/\xi_0^{3/2} Q_a^{1/2}$ by keeping t fixed and letting $a \to 0$. This is merely a convenient technical choice, and is by no means mandatory. The first-order equation is

$$\frac{\partial v_1}{\partial t} = \alpha h_0(t) \left[v_0 \frac{\partial^2 v_1}{\partial z^2} + 4 \frac{\partial v_0}{\partial z}\frac{\partial v_1}{\partial z} + \frac{\partial^2 v_0}{\partial z^2} v_1 \right] - \frac{v_0^2}{2\alpha h_0(t)} + \frac{h_1(t)}{h_0(t)} \frac{\partial v_0}{\partial t} \tag{3.9}$$

with the initial condition $v_1(z,0)=0$. For $|z| > h_0(t)$ we have the trivial solution $v_1(z,t)=0$, and for $|z| \le h_0(t)$, Eq. (3.9) for v_1 can be greatly simplified by the following variable transformations:

$$s = h_0^2(t) = (\xi_0^{3/2} Q_a^{1/2} t + a^{3/2})^{4/3} , \quad a^2 \le s < +\infty \tag{3.10}$$

or

$$\tau = \tfrac{1}{2} \ln s , \quad \delta \equiv \ln a \le \tau < +\infty , \tag{3.11}$$

and

$$x = \frac{z^2}{s} , \quad 0 \le x \le 1 . \tag{3.12}$$

Equation (3.9) then reduces to the form

$$\frac{\partial v_1}{\partial \tau} = \left[x(1-x) \frac{\partial^2 v_1}{\partial x^2} + \tfrac{1}{2}(1-5x) \frac{\partial v_1}{\partial x} - \tfrac{1}{2} v_1 \right] + f_1(x,\tau) + f_2(x,\tau) , \quad 0 \le x \le 1 \tag{3.13}$$

where

$$f_1(x,\tau) = -\frac{Q_a^{1/2} \xi_0^{3/2}}{48\alpha^3} e^{-\tau/2} (1-x)^2 \tag{3.14}$$

and

$$f_2(x,\tau) = -\frac{Q_a^{1/2} \xi_0^{3/2}}{12\alpha} e^{-3/2\tau} h_1(\tau)(1-5x) . \tag{3.15}$$

With the initial condition $v_1(\tau = \ln a, x) = 0$, the formal solution to Eq. (3.13) is given by

$$v_1(\tau,x) = \int_{\ln a}^{\tau} d\tau' \int_0^1 dx' G(\tau,x;\tau',x') [f_1(x',\tau') + f_2(x',\tau')] , \tag{3.16}$$

where G is the bounded Green's function satisfying

$$\frac{\partial G}{\partial \tau} - \left[x(1-x) \frac{\partial^2 G}{\partial x^2} + \tfrac{1}{2}(1-5x) \frac{\partial G}{\partial x} - \tfrac{1}{2} G \right] = \delta(\tau - \tau') \delta(x - x') . \tag{3.17}$$

The solution can be written as

$$G(\tau,x;\tau',x') = \sum_{n=0}^{\infty} \rho(x') \frac{1}{N_n^2} J_n(\alpha,\gamma,x) J_n(\alpha,\gamma,x') \times e^{-\lambda_n(\tau-\tau')} \Theta(\tau - \tau') , \tag{3.18}$$

where $\rho(x)$ is the weight function $\rho(x) = x^{-1/2}(1-x)$; $\alpha = \frac{3}{2}$, $\gamma = \frac{1}{2}$; the eigenvalues are $\lambda_n = n(n+\frac{3}{2}) + \frac{1}{2}$, $n = 0, 1, 2, \cdots$; $J_n(\alpha,\gamma,x)$ is the Jacobi polynomial of degree n; and N_n^2 is the normalization constant satisfying

$$\int_0^1 dx\, \rho(x) J_n(\alpha,\gamma,x) J_m(\alpha,\gamma,x) = N_n^2 \delta_{m,n} , \tag{3.19}$$

with $J_0 = 1$, $N_0^2 = \frac{4}{3}$ and $J_1 = 1 - 5x$, $N_1^2 = \frac{32}{21}$, etc. From the property of complete orthogonality of the eigenfunctions and the fact that $f_2(x',\tau') \propto J_1(x')$, then for $\tau \gg \ln a$ only the two combinations, i.e., J_0 and f_1, J_1 and f_2, contribute to the leading logarithmic divergence term to $O(\epsilon)$. We find that the singular part of the solution v_1 has the form

$$v_1^s = (v_1^s)_1 + (v_1^s)_2 , \tag{3.20}$$

where

$$(v_1^s)_1 = -\frac{Q_a^{1/2} \xi_0^{3/2}}{140\alpha^3} \frac{1}{s^{1/4}} \ln \left[\frac{s}{a^2} \right] \tag{3.21}$$

and

$$(v_1^s)_2 = -\frac{Q_a^{1/2} \xi_0^{3/2}}{24\alpha} \frac{1}{s^{3/2}} (1-5x) \int_{a^2}^{s} ds' \frac{h_1(s')}{s'^{1/4}} . \tag{3.22}$$

Since the zeroth-order solution is

$$v_0 = \frac{Q_a^{1/2} \xi_0^{3/2}}{6\alpha s^{1/4}} (1-x) \Theta(1-x) , \tag{3.23}$$

the bare-perturbation result to $O(\epsilon)$ is

$$v(s,z) = \frac{Q_a^{1/2} \xi_0^{3/2}}{6\alpha s^{1/4}} \left[1 - \frac{z^2}{s} - \epsilon \frac{3}{70\alpha^2} \ln \left[\frac{s}{a^2} \right] - \epsilon \frac{1}{4 s^{5/4}} \left[1 - \frac{5z^2}{s} \right] \int_{a^2}^{s} ds' \frac{h_1(s')}{s'^{1/4}} \right] + O(\epsilon) , \tag{3.24}$$

where $O(\epsilon)$ refers to finite terms regular in the limit $a \to 0$ which are unimportant for determining the anomalous dimensions, and only lead to finite corrections to the scaling functions. It is these terms to which we refer when we write $O(\epsilon)$ corrections in the following analysis. Using the fact that $v(z,s) = 0$ for values of $z > h(s)$, where the position is $z = h(s) = h_0(s) + \epsilon h_1(s) + \cdots$, we substitute the above expression in Eq. (3.24) and equate two sides of the equation order by order in ϵ, to obtain $h_0(s) = s^{1/4}$, with the singular part of $h_1(s)$ satisfying the integral equation

$$h_1^s(s) = -\frac{3}{140\alpha^2} s^{1/2} \ln \left[\frac{s}{a^2} \right] + \frac{1}{2 s^{3/4}} \int_{a^2}^{s} ds' \frac{h_1^s(s')}{s'^{1/4}} . \tag{3.25}$$

We solve by seeking a solution of the form

$$h_1^s(s) = A s^{1/2} \ln \left[\frac{s}{a^2} \right] , \tag{3.26}$$

giving $A=-1/28\alpha^2$. Thus,

$$h_1^s(s)=-\frac{1}{28\alpha^2}s^{1/2}\ln\left[\frac{s}{a^2}\right] . \tag{3.27}$$

Last, we rewrite $h(t)$ and $v(t,z)$ in the form

$$h(t)=(\xi_0^{3/2}Q_a^{1/2}t+a^{3/2})^{2/3}\times\left[1-\frac{\epsilon}{21\alpha^2}\ln\left[\frac{\xi_0^{3/2}Q_a^{1/2}t}{a^{3/2}}\right]\right]+O(\epsilon) \tag{3.28}$$

and

$$v(z,t)=\frac{Q_a^{1/2}\xi_0^{3/2}}{6\alpha(\xi_0^{3/2}Q_a^{1/2}t+a^{3/2})^{1/3}}\left[1-\frac{\epsilon}{21\alpha^2}\ln\left[\frac{\xi_0^{3/2}Q_a^{1/2}t}{a^{3/2}}\right]\right]-\frac{Q_a^{1/2}\xi_0^{3/2}z^2}{6\alpha(\xi_0^{3/2}Q_a^{1/2}t+a^{3/2})^{5/3}}\left[1+\frac{\epsilon}{21\alpha^2}\ln\left[\frac{\xi_0^{3/2}Q_a^{1/2}t}{a^{3/2}}\right]\right]+O(\epsilon) . \tag{3.29}$$

As expected, both $h(t)$ and $v(z,t)$ exhibit a leading singularity $\ln(\xi_0^{3/2}Q_a^{1/2}t/a^{3/2})$ in the limit $t/\xi_0^{3/2}Q_a^{1/2}a^{3/2}\to\infty$.

This divergence can be treated by regarding a as a regularization parameter. The singularity can be removed by introducing a renormalization constant $Z=Z(a/\mu,\epsilon)$, which absorbs the divergence in the limit $a\to 0$, order by order in ϵ. Hence, the renormalized quantities $h(t)$ and $v(z,t)$, which we shall denote by $h^R(t)$ and $v^R(z,t)$, respectively, remain finite even in the limit $a\to 0$. We replace Q_a by $Z(a/\mu,\epsilon)Q$ and Taylor expand Z as $Z=\sum_{n=0}^{\infty}a_n(a/\mu)\epsilon^n$ with $a_0=1$. The coefficients $a_n(n\geq 1)$ are determined order by order in ϵ in such a way that all the divergences in $h(t)$ and $v(z,t)$ are canceled out. Since

$$\ln Z=\ln[1+a_1\epsilon+O(\epsilon^2)]\approx a_1\epsilon+O(\epsilon^2) , \tag{3.30}$$

we have

$$h^R(t)=\xi_0Q^{1/3}t^{2/3}\left[1+\frac{a_1}{3}\epsilon+O(\epsilon^2)\right]\times\left[1-\frac{\epsilon}{21\alpha^2}\ln\left[\frac{\xi_0^{3/2}Q^{1/2}t}{a^{3/2}}\right]+O(\epsilon^2)\right]+O(\epsilon) , \tag{3.31}$$

By choosing $a_1(a/\mu)=-3/14\alpha^2\ln(C_1^{2/3}\mu/a)$, the divergence in the limit $a\to 0$ is removed. Here C_1 is an arbitrary constant which will in fact not appear in the final expression for the anomalous dimensions to $O(\epsilon)$. We assume that all such arbitrary constants, introduced if the renormalization process is pursued to higher order in ϵ, do not appear in the final results—an assumption known as perturbative renormalizability. We emphasize that this is very natural here, for the reasons outlined in Sec. II. Then we have

$$h_R(t)=\xi_0Q^{1/3}t^{2/3}\times\left[1-\frac{\epsilon}{21\alpha^2}\ln\left[\frac{C_1\xi_0^{3/2}Q^{1/2}t}{\mu^{3/2}}\right]\right]+O(\epsilon) , \tag{3.32}$$

which is independent of a and remains finite in the limit $a\to 0$. In the same way, the first term in Eq. (3.29) of v becomes

$$v^{R(1)}(z,t)=\frac{Q^{1/3}\xi_0}{6\alpha t^{1/3}}\left[1-\frac{\epsilon}{21\alpha^2}\ln\left[\frac{C_1\xi_0^{3/2}Q^{1/2}t}{\mu^{3/2}}\right]\right]+O(\epsilon) , \tag{3.33}$$

and the second term in Eq. (3.29) becomes

$$v^{R(2)}(z,t)=-\frac{z^2}{6a\xi_0Q^{1/3}t^{5/3}}\times\left[1+\frac{\epsilon}{21\alpha^2}\ln\left[\frac{C_1\xi_0^{3/2}Q^{1/2}t}{\mu^{3/2}}\right]\right]+O(\epsilon) . \tag{3.34}$$

Now we combine the renormalized perturbation series with the RG theory, as explained in Refs. [6] and [8], to obtain the final results for the long-time asymptotic behavior to $O(\epsilon)$:

$$h(t)\sim At^{2/3+\tilde\beta}+O(\epsilon) \tag{3.35}$$

and

$$v(z,t)\sim\frac{A}{6\alpha t^{1/3+\tilde\alpha}}\left[1-\left[\frac{z}{At^{2/3+(\beta-\tilde\alpha)/2}}\right]^2\right]\times\Theta\left[1-\left[\frac{z}{At^{2/3+(\beta-\tilde\alpha)/2}}\right]^2\right]+O(\epsilon) , \tag{3.36}$$

where we have introduced the anomalous dimensions

$$\beta=\tilde\beta=-\frac{\epsilon}{21\alpha^2}+O(\epsilon^2) \tag{3.37}$$

and

$$\tilde\alpha=-\beta=\frac{\epsilon}{21\alpha^2}+O(\epsilon^2) , \tag{3.38}$$

in agreement with the scaling law from our previous RG analysis. The phenomenological parameter A is a constant of integration of the original problem and has the formal value $A=\lim_{a\to 0}\xi_0^{1-3\tilde\alpha/2}Q_a^{1/3-\tilde\alpha/2}a^{3\tilde\alpha/2}$. Finally, we give the actual asymptotic form of the solution of turbulent energy with dissipation:

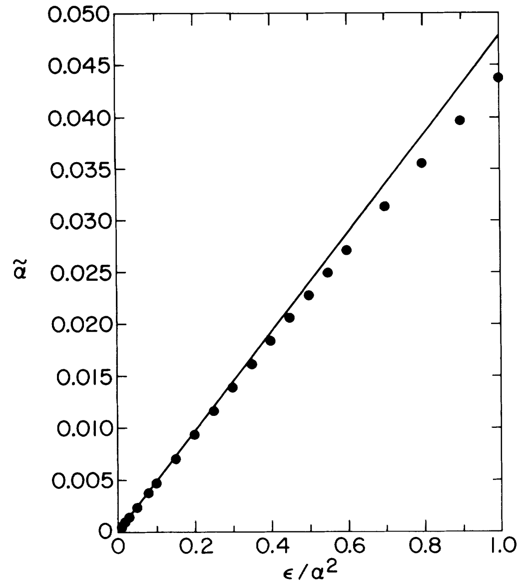

FIG. 1. The anomalous dimension $\tilde{\alpha}$ plotted as a function of ϵ/α^2. The full curve is the RG calculation to $O(\epsilon)$, whilst the points are calculated from numerical solution of the nonlinear eigenvalue problem, as explained in Ref. [14].

$$q(z,t)\sim\frac{A^2}{36\alpha^2 t^{2/3+2\tilde{\alpha}}}\left[1-\left[\frac{z}{At^{2/3-\tilde{\alpha}}}\right]^2\right]^2 \times\Theta\left[1-\left[\frac{z}{At^{2/3-\tilde{\alpha}}}\right]^2\right]+O(\epsilon)\ . \qquad (3.39)$$

In Fig. 1, the RG calculation of $\tilde{\alpha}$ is compared with the value of $\tilde{\alpha}$ obtained from a numerical solution of the nonlinear eigenvalue equation by using a shooting method. The two results are in good agreement for $0\le\epsilon/\alpha^2\le 0.5$. The relative error is less than 4.5% even for $\epsilon/\alpha^2=0.5$.

In conclusion, we have used general renormalization-group arguments to show that the long-time behavior of the propagation of a turbulent layer in the presence of dissipation ϵ is governed by a similarity solution with anomalous exponents that cannot be determined by dimensional analysis. We have used renormalization-group methods, originally developed in quantum field theory, in conjunction with perturbation theory to calculate these exponents to $O(\epsilon)$. Although we are unable to make any definite statements about the convergence of the expansion that we have assumed here, it should be noted that in the problem of nonlinear diffusion in an elastoplastic porous medium, our renormalization-group expansion for the anomalous dimension has been proven rigorously to be analytic [23].

Note added in proof. The existence and uniqueness of solutions to Eq. (1.5) have been rigorously proved, and the convergence to a self-similar solution with anomalous dimensions established by S. Kamin and J. L. Vazquez (unpublished).

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